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ON CERTAIN INEQUALITIES RELATED TO THE SEITZ INEQUALITY

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ABSTRACT. In this paper, we investigate the monotonicity of difference results from the G. Seitz inequality. An application is given, with some resulting inequalities.

Key words and phrases: G. Seitz inequality, Convex function, Difference, Monotonicity, Exponential convex.

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1. INTRODUCTION

For a given positive integer $n \geq 2$, let $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$, $U = (u_1, u_2, \dots, u_n)$ and $Z = (z_1, z_2, \dots, z_n)$ be known sequences of real numbers, and let $t_i > 0$ $(i = 1, 2, \dots, n)$, $T_j = \sum_{i=1}^{j} t_i$ $(j = 1, 2, \dots, n)$ and a_{ij} $(i, j = 1, 2, \dots, n)$ be known real numbers. Define the functions A, J, C, W and G by

$$A(n) \triangleq \sum_{i=1}^{n} x_i - n \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \quad (x_i > 0, i = 1, 2, \dots, n),$$

$$J(n) \triangleq \sum_{i=1}^{n} t_i f(v_i) - T_n f \left(\frac{1}{T_n} \sum_{i=1}^{n} t_i v_i \right),$$

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where f is convex function on the interval I and $v_i \in I (i = 1, 2, \dots, n)$,

$$C(n) \triangleq \left[\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)\right]^{\frac{1}{2}} - \sum_{i=1}^{n} x_i y_i,$$

$$W(n) \triangleq T_n \sum_{i=1}^{n} t_i x_i z_i - \left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{i=1}^{n} t_i z_i\right),$$

and

$$G(n) \triangleq \left(\sum_{i,j=1}^{n} a_{ij} x_i z_j\right)\left(\sum_{i,j=1}^{n} a_{ij} y_i u_j\right) - \left(\sum_{i,j=1}^{n} a_{ij} x_i u_j\right)\left(\sum_{i,j=1}^{n} a_{ij} y_i z_j\right).$$

Rade investigated the monotonicity of difference for $A - G$ mean inequality, and obtained the following inequality [2]

$$A(n) \geq A(n-1). \tag{1.1}$$

P. M. Vasić and J. E. Pečarić generalized inequality (1.1) to convex functions, and obtained the following inequality [4, 6]

$$J(n) \geq J(n-1). \tag{1.2}$$

Recently the first author and Xu Zhang studied inequality (1.2) in depth, and obtained some inequalities. L.-C. Wang also obtained some applications, one of them is the following inequality [8]

$$C(n) \geq C(n-1). \tag{1.3}$$

Inequality (1.3) resulted from the Cauchy inequality

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \geq \left(\sum_{i=1}^{n} x_i y_i\right)^2. \tag{1.4}$$

In [7], L.-C. Wang proved the following inequality

$$W(n) \geq W(n-1), \tag{1.5}$$

with X and Z both increasing or both decreasing. If one of X or Z is increasing and the other decreasing, then the inequality (1.5) reverses.

Inequality (1.5) resulted from the following Chebyshev inequality

$$T_n \sum_{i=1}^{n} t_i x_i z_i \geq \left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{i=1}^{n} t_i z_i\right), \tag{1.6}$$

with X and Z both increasing or both decreasing. If one of X or Z is increasing and the other decreasing, then the inequality (1.6) reverses.

Assume that $i, j, r, s \in \mathbb{N}$ such that $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$, we have

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \begin{vmatrix} z_r & z_s \\ u_r & u_s \end{vmatrix} \geq 0 \tag{1.7}$$

and

$$\begin{vmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{vmatrix} \geq 0. \tag{1.8}$$

When both (1.7) and (1.8) are true, the following inequality by G. Seitz [1] holds:

$$\frac{\sum_{i,j=1}^{n} a_{ij}x_i z_j}{\sum_{i,j=1}^{n} a_{ij}x_i u_j} \geq \frac{\sum_{i,j=1}^{n} a_{ij}y_i z_j}{\sum_{i,j=1}^{n} a_{ij}y_i u_j}. \tag{1.9}$$

If

$$X = Z, \quad Y = U \quad \text{and} \quad a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (i,j = 1,2,\dots,n), \tag{1.10}$$

then inequality (1.9) changes into (1.4). If

$$Y = U = (1,1,\dots,1) \quad \text{and} \quad a_{ij} = \begin{cases} t_i & i = j \\ 0 & i \neq j \end{cases} \quad (i,j = 1,2,\dots,n), \tag{1.11}$$

then inequality (1.9) changes into (1.6).

In this paper, we investigate inequality (1.9) in depth, obtaining the following main result.

Theorem 1.1. *If both inequalities (1.7) and (1.8) are true, then we have*

$$G(n) \geq G(n-1). \tag{1.12}$$

Remark 1.2. If we put (1.10) and (1.11) into (1.12), then (1.12) becomes (1.3) and (1.5), respectively. Hence, (1.12) is an extension of (1.3) and (1.5).

2. PROOF OF THEOREM 1.1

Using

$$\left(a_{ij}x_i z_j\right)\left(a_{in}y_i u_n\right) = \left(a_{ij}y_i z_j\right)\left(a_{in}x_i u_n\right) \quad (i,j = 1,2,\dots,n-1),$$

$$\left(a_{ij}y_i u_j\right)\left(a_{in}x_i z_n\right) = \left(a_{ij}x_i u_j\right)\left(a_{in}y_i z_n\right) \quad (i,j = 1,2,\dots,n-1),$$

and (1.7) – (1.8), we have

$$\begin{aligned} &\sum_{i,j=1}^{n-1} a_{ij}x_i z_j \sum_{i=1}^{n-1} a_{in}y_i u_n - \sum_{i,j=1}^{n-1} a_{ij}y_i z_j \sum_{i=1}^{n-1} a_{in}x_i u_n \\ &\qquad + \sum_{i,j=1}^{n-1} a_{ij}y_i u_j \sum_{i=1}^{n-1} a_{in}x_i z_n - \sum_{i,j=1}^{n-1} a_{ij}x_i u_j \sum_{i=1}^{n-1} a_{in}y_i z_n \\ &= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}x_i z_j \sum_{k=1}^{n-1} a_{kn}y_k u_n - \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}y_i z_j \sum_{k=1}^{n-1} a_{kn}x_k u_n \\ &\qquad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}y_i u_j \sum_{k=1}^{n-1} a_{kn}x_k z_n - \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} a_{ij}x_i u_j \sum_{k=1}^{n-1} a_{kn}y_k z_n \end{aligned} \tag{2.1}$$

$$= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1,k\neq i}^{n-1} a_{ij}a_{kn}\Big(x_iz_jy_ku_n + x_kz_ny_iu_j - y_iz_jx_ku_n - x_iu_jy_kz_n\Big)$$

$$= \sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-2}\sum_{k=2,i<k}^{n-1} + \sum_{i=2}^{n-1}\sum_{k=1,i>k}^{n-2}\right) a_{ij}a_{kn}\Big(x_iz_jy_ku_n + x_kz_ny_iu_j - y_iz_jx_ku_n - x_iu_jy_kz_n\Big)$$

$$= \sum_{j=1}^{n-1}\sum_{i=1}^{n-2}\sum_{k=2,i<k}^{n-1} a_{ij}a_{kn}\Big(x_iz_jy_ku_n + x_kz_ny_iu_j - y_iz_jx_ku_n - x_iu_jy_kz_n\Big)$$
$$+ \sum_{j=1}^{n-1}\sum_{k=2}^{n-1}\sum_{i=1,k>i}^{n-2} a_{kj}a_{in}\Big(x_kz_jy_iu_n + x_iz_ny_ku_j - y_kz_jx_iu_n - x_ku_jy_iz_n\Big)$$

$$= \sum_{j=1}^{n-1}\sum_{1\le i<k<n}\Big(a_{ij}a_{kn} - a_{kj}a_{in}\Big)\Big(x_iz_jy_ku_n + x_kz_ny_iu_j - y_iz_jx_ku_n - x_iu_jy_kz_n\Big)$$

$$= \sum_{j=1}^{n-1}\sum_{1\le i<k<n}\begin{vmatrix} a_{ij} & a_{in} \\ a_{kj} & a_{kn}\end{vmatrix}\begin{vmatrix} x_i & x_k \\ y_i & y_k\end{vmatrix}\begin{vmatrix} z_j & z_n \\ u_j & u_n\end{vmatrix} \ge 0.$$

Using

$$\Big(a_{ij}x_iz_j\Big)\Big(a_{nj}y_nu_j\Big) = \Big(a_{ij}x_iu_j\Big)\Big(a_{nj}y_nz_j\Big) \quad (i,j = 1,2,\dots,n-1),$$

$$\Big(a_{ij}y_iu_j\Big)\Big(a_{nj}x_nz_j\Big) = \Big(a_{ij}y_iz_j\Big)\Big(a_{nj}x_nu_j\Big) \quad (i,j = 1,2,\dots,n-1),$$

(1.7) – (1.8) and the same method as in the proof of (2.1), we obtain

$$\sum_{i,j=1}^{n-1} a_{ij}x_iz_j\sum_{j=1}^{n-1} a_{nj}y_nu_j - \sum_{i,j=1}^{n-1} a_{ij}x_iu_j\sum_{j=1}^{n-1} a_{nj}y_nz_j \tag{2.2}$$
$$+ \sum_{i,j=1}^{n-1} a_{ij}y_iu_j\sum_{j=1}^{n-1} a_{nj}x_nz_j - \sum_{i,j=1}^{n-1} a_{ij}y_iz_j\sum_{j=1}^{n-1} a_{nj}x_nu_j$$
$$= \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{k=1,k\neq j}^{n-1} a_{ij}a_{nk}\Big(x_iz_jy_nu_k + y_iu_jx_nz_k - x_iu_jy_nz_k - y_iz_jx_nu_k\Big)$$
$$= \sum_{i=1}^{n-1}\sum_{1\le j<k<n}\begin{vmatrix} a_{ij} & a_{ik} \\ a_{nj} & a_{nk}\end{vmatrix}\begin{vmatrix} x_i & x_n \\ y_i & y_n\end{vmatrix}\begin{vmatrix} z_j & z_k \\ u_j & u_k\end{vmatrix} \ge 0.$$

Using

$$\Big(a_{in}y_iu_n\Big)\Big(a_{jn}x_jz_n\Big) = \Big(a_{in}y_iz_n\Big)\Big(a_{jn}x_ju_n\Big) \quad (i,j = 1,2,\dots,n)$$

and

$$\Big(a_{ni}y_nu_i\Big)\Big(a_{nj}x_nz_j\Big) = \Big(a_{ni}y_nz_i\Big)\Big(a_{nj}x_nu_j\Big) \quad (i,j = 1,2,\dots,n-1),$$

we obtain

$$\sum_{i=1}^{n} a_{in}y_iu_n\sum_{j=1}^{n} a_{jn}x_jz_n - \sum_{i=1}^{n} a_{in}y_iz_n\sum_{j=1}^{n} a_{jn}x_ju_n = 0 \tag{2.3}$$

and

$$\sum_{i=1}^{n-1} a_{ni} y_n u_i \sum_{j=1}^{n-1} a_{nj} x_n z_j - \sum_{i=1}^{n-1} a_{ni} y_n z_i \sum_{j=1}^{n-1} a_{nj} x_n u_j = 0, \tag{2.4}$$

respectively.

Using

$$\Big(a_{nn} y_n u_n\Big)\Big(a_{nj} x_n z_j\Big) = \Big(a_{nj} y_n z_j\Big)\Big(a_{nn} x_n u_n\Big) \quad (j = 1, 2, \dots, n-1),$$

$$\Big(a_{nj} y_n u_j\Big)\Big(a_{nn} x_n z_n\Big) = \Big(a_{nn} y_n z_n\Big)\Big(a_{nj} x_n u_j\Big) \quad (j = 1, 2, \dots, n-1),$$

and (1.7) – (1.8), we have

$$\begin{aligned}
&\left(\sum_{i=1}^{n} a_{in} y_i u_n \sum_{j=1}^{n-1} a_{nj} x_n z_j - \sum_{j=1}^{n-1} a_{nj} y_n z_j \sum_{i=1}^{n} a_{in} x_i u_n\right) \\
&\quad + \left(\sum_{j=1}^{n-1} a_{nj} y_n u_j \sum_{i=1}^{n} a_{in} x_i z_n - \sum_{i=1}^{n} a_{in} y_i z_n \sum_{j=1}^{n-1} a_{nj} x_n u_j\right) \\
&\quad + \left(\sum_{i,j=1}^{n-1} a_{ij} a_{nn} x_i z_j y_n u_n - \sum_{i,j=1}^{n-1} a_{ij} a_{nn} y_i z_j x_n u_n\right) \\
&\quad + \left(\sum_{i,j=1}^{n-1} a_{ij} a_{nn} y_i u_j x_n z_n - \sum_{i,j=1}^{n-1} a_{ij} a_{nn} x_i u_j y_n z_n\right) \\
&= \left(\sum_{i=1}^{n-1} a_{in} y_i u_n \sum_{j=1}^{n-1} a_{nj} x_n z_j - \sum_{j=1}^{n-1} a_{nj} y_n z_j \sum_{i=1}^{n-1} a_{in} x_i u_n\right) \\
&\quad + \left(\sum_{j=1}^{n-1} a_{nj} y_n u_j \sum_{i=1}^{n-1} a_{in} x_i z_n - \sum_{i=1}^{n-1} a_{in} y_i z_n \sum_{j=1}^{n-1} a_{nj} x_n u_j\right) \\
&\quad + \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} a_{nn} x_i z_j y_n u_n - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} a_{nn} y_i z_j x_n u_n\right) \\
&\quad + \left(\sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} a_{nn} y_i u_j x_n z_n - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} a_{nn} x_i u_j y_n z_n\right) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{in} a_{nj} \Big(y_i u_n x_n z_j + x_i z_n y_n u_j - x_i u_n y_n z_j - y_i z_n x_n u_j\Big) \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} a_{ij} a_{nn} \Big(x_i z_j y_n u_n + x_n z_n y_i u_j - y_i z_j x_n u_n - x_i u_j y_n z_n\Big) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \Big(a_{ij} a_{nn} - a_{in} a_{nj}\Big)\Big(x_i z_j y_n u_n + x_n z_n y_i u_j - y_i z_j x_n u_n - x_i u_j y_n z_n\Big) \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \begin{vmatrix} a_{ij} & a_{in} \\ a_{nj} & a_{nn} \end{vmatrix} \begin{vmatrix} x_i & x_n \\ y_i & y_n \end{vmatrix} \begin{vmatrix} z_j & z_n \\ u_j & u_n \end{vmatrix} \geq 0.
\end{aligned} \tag{2.5}$$

By (2.1)-(2.5) and definition of $G(n)$, we have

$$\begin{aligned}
G(n) - G(n-1) &= \left(\sum_{i,j=1}^{n-1} a_{ij}x_iz_j + \sum_{i=1}^{n} a_{in}x_iz_n + \sum_{j=1}^{n-1} a_{nj}x_nz_j\right) \\
&\qquad\qquad \times \left(\sum_{i,j=1}^{n-1} a_{ij}y_iu_j + \sum_{i=1}^{n} a_{in}y_iu_n + \sum_{j=1}^{n-1} a_{nj}y_nu_j\right) \\
&\qquad - \left(\sum_{i,j=1}^{n-1} a_{ij}x_iu_j + \sum_{i=1}^{n} a_{in}x_iu_n + \sum_{j=1}^{n-1} a_{nj}x_nu_j\right) \\
&\qquad\qquad \times \left(\sum_{i,j=1}^{n-1} a_{ij}y_iz_j + \sum_{i=1}^{n} a_{in}y_iz_n + \sum_{j=1}^{n-1} a_{nj}y_nz_j\right) \\
&\qquad - \left(\sum_{i,j=1}^{n-1} a_{ij}x_iz_j \sum_{i,j=1}^{n-1} a_{ij}y_iu_j - \sum_{i,j=1}^{n-1} a_{ij}x_iu_j \sum_{i,j=1}^{n-1} a_{ij}y_iz_j\right) \\
&= \left(\sum_{i,j=1}^{n-1} a_{ij}x_iz_j \sum_{i=1}^{n-1} a_{in}y_iu_n - \sum_{i,j=1}^{n-1} a_{ij}y_iz_j \sum_{i=1}^{n-1} a_{in}x_iu_n\right) \\
&\qquad + \left(\sum_{i,j=1}^{n-1} a_{ij}a_{nn}x_iz_jy_nu_n - \sum_{i,j=1}^{n-1} a_{ij}a_{nn}y_iz_jx_nu_n\right) \\
&\qquad + \left(\sum_{i,j=1}^{n-1} a_{ij}y_iu_j \sum_{i=1}^{n-1} a_{in}x_iz_n - \sum_{i,j=1}^{n-1} a_{ij}x_iu_j \sum_{i=1}^{n-1} a_{in}y_iz_n\right) \\
&\qquad + \left(\sum_{i,j=1}^{n-1} a_{ij}a_{nn}y_iu_jx_nz_n - \sum_{i,j=1}^{n-1} a_{ij}a_{nn}x_iu_jy_nz_n\right) \\
&\qquad + \left(\sum_{i,j=1}^{n-1} a_{ij}x_iz_j \sum_{j=1}^{n-1} a_{nj}y_nu_j - \sum_{i,j=1}^{n-1} a_{ij}x_iu_j \sum_{j=1}^{n-1} a_{nj}y_nz_j\right) \\
&\qquad + \left(\sum_{i,j=1}^{n-1} a_{ij}y_iu_j \sum_{j=1}^{n-1} a_{nj}x_nz_j - \sum_{i,j=1}^{n-1} a_{ij}y_iz_j \sum_{j=1}^{n-1} a_{nj}x_nu_j\right) \\
&\qquad + \left(\sum_{i=1}^{n} a_{in}y_iu_n \sum_{j=1}^{n} a_{jn}x_jz_n - \sum_{i=1}^{n} a_{in}y_iz_n \sum_{j=1}^{n} a_{jn}x_ju_n\right) \\
&\qquad + \left(\sum_{i=1}^{n-1} a_{ni}y_nu_i \sum_{j=1}^{n-1} a_{nj}x_nz_j - \sum_{i=1}^{n-1} a_{ni}y_nz_i \sum_{j=1}^{n-1} a_{nj}x_nu_j\right) \\
&\qquad + \left(\sum_{i=1}^{n} a_{in}y_iu_n \sum_{j=1}^{n-1} a_{nj}x_nz_j - \sum_{j=1}^{n-1} a_{nj}y_nz_j \sum_{i=1}^{n} a_{in}x_iu_n\right) \\
&\qquad + \left(\sum_{j=1}^{n-1} a_{nj}y_nu_j \sum_{i=1}^{n} a_{in}x_iz_n - \sum_{i=1}^{n} a_{in}y_iz_n \sum_{j=1}^{n-1} a_{nj}x_nu_j\right) \\
&\geq 0,
\end{aligned}$$

i.e., inequality (1.12) is true. This completes the proof of theorem .

3. APPLICATIONS

Let E be a convex subset of an arbitrary real linear space $\mathbb{K}$, and let $f : E \mapsto (0, +\infty)$. f is an exponential convex function on E, if and only if

$$f\Big(tu + (1-t)v\Big) \le f^t(u) f^{1-t}(v) \tag{3.1}$$

for any $u, v \in E$ and any $t \in [0, 1]$. f is an exponential concave function on E, if and only if the inequality (3.1) reverses (see [4]).

For any $u, v \in E (u \neq v)$ and $\alpha_{ki}, \beta_{ki} \in [0, 1]$, we let $x_{ki} = \alpha_{ki} u + (1 - \alpha_{ki})v$ and $y_{ki} = \beta_{ki} u + (1 - \beta_{ki})v \quad (k = 1, 2; i = 1, 2, \ldots, n; n > 2)$. Define a function L by

$$L(n) = \sum_{i,j=1}^{n} a_{ij} f(x_{1i}) f(x_{2j}) \sum_{i,j=1}^{n} a_{ij} f(y_{1i}) f(y_{2j}) - \sum_{i,j=1}^{n} a_{ij} f(x_{1i}) f(y_{2j}) \sum_{i,j=1}^{n} a_{ij} f(y_{1i}) f(x_{2j}).$$

Proposition 3.1. *Let f be an exponential convex (or concave) function on E and inequality (1.8) be true. For $k = 1, 2$ and every pair of positive integers i and j such that $1 \le i < j \le n$, if*

$$\alpha_{ki} \le \beta_{ki} \le \alpha_{kj} \qquad \textit{and} \qquad \alpha_{kj} - \alpha_{ki} = \beta_{kj} - \beta_{ki}, \tag{3.2}$$

then we have

$$L(n) \ge L(n-1). \tag{3.3}$$

Proof. (1) Suppose f is an exponential convex function on E. For $k = 1, 2$ and $1 \le i < j \le n$, from (3.2), we have $\beta_{kj} = \alpha_{kj} + \beta_{ki} - \alpha_{ki} \ge \alpha_{kj}$.

Case 1. When $\alpha_{ki} < \beta_{ki} \le \alpha_{kj} < \beta_{kj}$, we take $t = \frac{\beta_{ki} - \alpha_{ki}}{\beta_{kj} - \alpha_{ki}}$, then $1 - t = \frac{\beta_{kj} - \beta_{ki}}{\beta_{kj} - \alpha_{ki}}$. Hence, we have

$$t y_{kj} + (1-t) x_{ki} = \beta_{ki} u + (1 - \beta_{ki}) v = y_{ki}. \tag{3.4}$$

From (3.1) and (3.4), we have

$$f(y_{ki}) \le f^t(y_{kj}) f^{1-t}(x_{ki}). \tag{3.5}$$

From (3.2), we get the other form of t and $1-t$: $t = \frac{\beta_{kj} - \alpha_{kj}}{\beta_{kj} - \alpha_{ki}}$ and $1 - t = \frac{\alpha_{kj} - \alpha_{ki}}{\beta_{kj} - \alpha_{ki}}$. Then we have

$$(1-t) y_{kj} + t x_{ki} = \alpha_{kj} u + (1 - \alpha_{kj}) v = x_{kj}. \tag{3.6}$$

From (3.1) and (3.6), we have

$$f(x_{kj}) \le f^{1-t}(y_{kj}) f^t(x_{ki}). \tag{3.7}$$

From (3.5) and (3.7), we obtain

$$\begin{vmatrix} f(x_{ki}) & f(x_{kj}) \\ f(y_{ki}) & f(y_{kj}) \end{vmatrix} \ge 0. \tag{3.8}$$

Case 2. When $\alpha_{ki} = \beta_{ki}$ (or $\alpha_{kj} = \beta_{kj}$), by (3.2), then we have $\alpha_{kj} = \beta_{kj}$ (or $\alpha_{ki} = \beta_{ki}$). Hence, the equality of (3.8) holds.

For any $1 \le i < j \le n$ and any $1 \le r < s \le n$, by (3.8), we obtain

$$\begin{vmatrix} f(x_{1i}) & f(x_{1j}) \\ f(y_{1i}) & f(y_{1j}) \end{vmatrix} \begin{vmatrix} f(x_{2r}) & f(x_{2s}) \\ f(y_{2r}) & f(y_{2s}) \end{vmatrix} \ge 0. \tag{3.9}$$

(2) Let f be an exponential concave function on E. Then (3.5), (3.7) and (3.8) reverse. Hence, (3.9) is still valid.

Replacing x_i, y_i, z_i and u_i in Theorem 1.1 with $f(x_{1i})$, $f(y_{1i})$, $f(x_{2i})$ and $f(y_{2i})(i = 1, 2, \ldots, n)$, respectively, we obtain (3.3). This completes the proof of Proposition 3.1. □

Remark 3.2. In Proposition 3.1, when E is a real interval I, we only need

$$x_{ki} \leq y_{ki} \leq x_{kj} \quad \text{and} \quad x_{kj} - x_{ki} = y_{kj} - y_{ki},$$

where $k = 1, 2, i, j$ are every pair of positive integers such that $1 \leq i < j \leq n$, $x_{ki}, x_{kj}, y_{ki}, y_{kj} \in I$.

In order to verify Proposition 3.4, the following lemma is necessary.

Lemma 3.3. *Let $c, d : [a, b] \mapsto \mathbb{R}$ $(b > a)$ be the monotonic functions, both increasing or both decreasing. Furthermore, let $q : [a, b] \mapsto (0, +\infty)$ be an integrable function. Then*

$$\int_a^b q(x)c(x)dx \int_a^b q(x)d(x)dx \leq \int_a^b q(x)dx \int_a^b q(x)c(x)d(x)dx. \tag{3.10}$$

If one of the functions of c or d is increasing and the other decreasing, then the inequality (3.10) reverses. (see [2, 3]*).*

Let $p : [a, b] \mapsto (0, +\infty)$ be continuous, $g : [a, b] \mapsto (1, +\infty)$ be monotonic continuous. Define a function M by

$$\begin{aligned} M(n) = \sum_{i,j}^{n} a_{ij} h^{(k+i)}(x) h^{(m+j)}(x) \sum_{i,j}^{n} a_{ij} h^{(l+i)}(x) h^{(w+j)}(x) \\ - \sum_{i,j}^{n} a_{ij} h^{(k+i)}(x) h^{(w+j)}(x) \sum_{i,j}^{n} a_{ij} h^{(l+i)}(x) h^{(m+j)}(x), \end{aligned}$$

where $k, l, m, w \in \mathbb{N}$, $i, j = 1, 2, \ldots, n$ and

$$h : \mathbb{R} \mapsto \mathbb{R}^+, \qquad h(x) = \int_a^b p(t)\Big(g(t)\Big)^x dt \quad \text{(see [5])}. \tag{3.11}$$

Proposition 3.4. *Let the inequalities in (1.8) hold. If $k < l$, $m < w$ or $k > l$, $m > w$, then we have*

$$M(n) \geq M(n-1). \tag{3.12}$$

Proof. For (3.11), by continuity of p and g, we may change the order of derivation and integration. By direct computation, we get

$$h^{(n)}(x) = \int_a^b p(t) \left(g(t)\right)^x \left(\ln g(t)\right)^n dt. \tag{3.13}$$

For every pair of integers i, j such that $1 \leq i < j \leq n$, when $k < l$, replace q, c and d in Lemma 3.3 by $p(t)\left(g(t)\right)^x \left(\ln g(t)\right)^{k+i}$, $\left(\ln g(t)\right)^{j-i}$ and $\left(\ln g(t)\right)^{l-k}$, respectively. Using (3.13), we get

$$h^{(k+i)}(x) h^{(l+j)}(x) \geq h^{(k+j)}(x) h^{(l+i)}(x). \tag{3.14}$$

By (3.14), we have

$$\begin{vmatrix} h^{(k+i)}(x) & h^{(k+j)}(x) \\ h^{(l+i)}(x) & h^{(l+j)}(x) \end{vmatrix} \geq 0. \tag{3.15}$$

Similarly we obtain

$$\begin{vmatrix} h^{(m+r)}(x) & h^{(m+s)}(x) \\ h^{(w+r)}(x) & h^{(w+s)}(x) \end{vmatrix} \geq 0, \tag{3.16}$$

where r, s are pair of integer such that $1 \leq r < s \leq n$ and $m < w$.

Replacing x_i, y_i, z_i and u_i in Theorem 1.1 by $h^{(k+i)}(x)$, $h^{(l+i)}(x)$, $h^{(m+i)}(x)$ and $h^{(w+i)}(x)(i = 1, 2, \ldots, n)$, respectively, we obtain (3.12).

By Lemma 3.3, when $k > l$ and $m > w$, both (3.15) and (3.16) reverse. Hence, the product on the left for both (3.15) and (3.16) is still nonnegative, hence, by Theorem 1.1, (3.12) is also satisfied.

This completes the proof of Proposition 3.4. □

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