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# INEQUALITIES FOR WALSH POLYNOMIALS WITH SEMI-MONOTONE AND SEMI-CONVEX COEFFICIENTS

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ABSTRACT. Using the concept of majorant sequences (see [4, ch. XXI], [5], [7], [8]) some new inequalities for Walsh polynomials with complex semi-monotone, complex semi-convex, complex monotone and complex convex coefficients are given.

*Key words and phrases:* Petrovic inequality, Walsh polynomial, Complex semi-convex coefficients, Complex convex coefficients, Complex semi-monotone coefficients, Complex monotone coefficients, Fine inequality.

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## 1. INTRODUCTION AND PRELIMINARIES

We consider the Walsh orthonormal system $\{w_n(x)\}_{n=0}^{\infty}$ defined on $[0,1)$ in the Paley enumeration. Thus $w_0(x) \equiv 1$ and for each positive integer with dyadic development

$$n = \sum_{i=1}^{p} 2^{\nu_i}, \quad \nu_1 > \nu_2 > \cdots > \nu_p \geq 0,$$

we have

$$w_n(x) = \prod_{i=1}^{p} r_{\nu_i}(x),$$

where $\{r_n(x)\}_{n=0}^{\infty}$ denotes the Rademacher system of functions defined by (see, e.g. [1, p. 60], [3, p. 9-10])

$$r_\nu(x) = \operatorname{sign} \sin 2^\nu \pi(x) \quad (\nu = 0, 1, 2, \dots;\ 0 \leq x < 1).$$

In this paper we shall consider the Walsh polynomials $\sum_{k=n}^{m} \lambda_k w_k(x)$ with complex-valued coefficients $\{\lambda_k\}$.

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Let $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1} = \lambda_n - 2\lambda_{n+1} + \lambda_{n+2}$, for all $n = 1, 2, 3 \dots$.

Petrović [6] proved the following complementary triangle inequality for a sequence of complex numbers $\{z_1, z_2, \dots, z_n\}$.

**Theorem A.** *Let $\alpha$ be a real number and $0 < \theta < \frac{\pi}{2}$. If $\{z_1, z_2, \dots, z_n\}$ are complex numbers such that $\alpha - \theta \le \arg z_\nu \le \alpha + \theta$, $\nu = 1, 2, \dots, n$, then*

$$\left|\sum_{\nu=1}^{n} z_\nu\right| \ge (\cos\theta)\sum_{\nu=1}^{n} |z_\nu|.$$

For $0 < \theta < \frac{\pi}{2}$ denote by $K(\theta)$ the cone $K(\theta) = \{z : |\arg z| \le \theta\}$.

Let $\{b_k\}$ be a positive nondecreasing sequence. The following definitions are given in [7] and [8]. The sequence of complex numbers $\{u_k\}$ is said to be **complex semi$-$monotone** if there exists a cone $K(\theta)$ such that $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$ or $\Delta(u_k b_k) \in K(\theta)$. For $b_k = 1$, the sequence $\{u_k\}$ shall be called a **complex monotone** sequence. On the other hand, the sequence $\{u_k\}$ is said to be **complex semi$-$convex** if there exists a cone $K(\theta)$, such that $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$ or $\Delta^2(u_k b_k) \in K(\theta)$. For $b_k = 1$, the sequence $\{u_k\}$ shall be called a **complex convex sequence**.

The following two Theorems were proved by Tomovski in [7] and [8].

**Theorem B** ([7])**.** *Let $\{z_k\}$ be a sequence such that $|\sum_{k=n}^{m} z_k| \le A$, $(\forall n, m \in \mathbb{N}, m > n)$, where $A$ is a positive number.*

(i) *If $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \le A\left[\left(1 + \frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_n|\right], \quad (\forall n, m \in \mathbb{N}, m > n).$$

(ii) *If $\Delta(u_k b_k) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \le A\left[\left(1 + \frac{1}{\cos\theta}\right)|u_n| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_m|\right], \quad (\forall n, m \in \mathbb{N}, m > n).$$

**Theorem C** ([8])**.** *Let $A = \max\limits_{n \le p \le q \le m}\left|\sum\limits_{j=p}^{q}\sum\limits_{k=i}^{j} z_k\right|$.*

(i) *If $\{u_k\}$ is a sequence of complex numbers such that $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \le A\left[|u_m| + b_m\left(1 + \frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_m}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right],$$
$$(\forall n, m \in \mathbb{N}, m > n).$$

(ii) *If $\{u_k\}$ is a sequence of complex numbers such that $\Delta^2(u_k b_k) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \le A\left[|u_n| + b_n^{-1}\left(1 + \frac{1}{\cos\theta}\right)\left(|\Delta(u_n b_n)| + |\Delta(u_{m-1} b_{m-1})|\right)\right],$$
$$(\forall n, m \in \mathbb{N}, m > n).$$

Using the concept of majorant sequences we shall give some estimates for Walsh polynomials with complex semi-monotone, complex monotone, complex semi-convex and complex convex coefficients.

## 2. MAIN RESULTS

For the main results we require the following Lemma.

**Lemma 2.1.** *For all $p, q, r \in \mathbb{N}, p < q$ the following inequalities hold:*

(i) $\left|\sum_{k=p}^{q} \omega_k(x)\right| \le \frac{2}{x}, 0 < x < 1.$

(ii) $\left|\sum_{j=p}^{q}\sum_{k=l}^{j} \omega_k(x)\right| \le \begin{cases} \frac{2(q-p+1)}{x} = C_1(p,q,x) : 0 < x < 1 \\ \frac{8}{x(x-2^{-r})} + \frac{8}{x^2} + \frac{2(q-p+1)}{x} + 1 = C_2(p,q,r,x) : x \in (2^{-r}, 2^{-r+1}) \end{cases}$

*Proof.* (i) Let $D_q(x) = \sum_{i=0}^{q-1} w_i(x)$ be the Dirichlet kernel. Then it is known that (see [3, p. 28]) $|D_q(x)| \le \frac{1}{x}, 0 < x < 1$. Hence

$$\left|\sum_{k=p}^{q} w_k(x)\right| = |D_{q+1}(x) - D_p(x)| \le |D_{q+1}(x)| + |D_p(x)| \le \frac{2}{x}.$$

(ii) By (i) we get

$$\left|\sum_{j=p}^{q}\sum_{k=l}^{j} w_k(x)\right| \le \sum_{j=p}^{q}\left|\sum_{k=l}^{j} w_k(x)\right| \le \frac{2(q-p+1)}{x}, \quad 0 < x < 1.$$

Let $F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x)$ be the Fejer kernel. Applying Fine's inequality (see [2])

$$(n+1)F_n(x) < \frac{4}{x(x-2^{-r})} + \frac{4}{x^2}, \quad x \in (2^{-r}, 2^{-r+1}),$$

we get

$$\begin{aligned} \left|\sum_{j=p}^{q}\sum_{k=l}^{j} w_k(x)\right| &= \left|\sum_{j=p}^{q}(D_{j+1}(x) - D_l(x))\right| \\ &\le \left|\sum_{j=p}^{q} D_{j+1}(x)\right| + \frac{q-p+1}{x} \\ &\le |(q+1)F_q(x)| + |D_{q+1}(x)| + |D_0(x)| + |pF_{p-1}(x)| + \frac{q-p+1}{x} \\ &< \frac{8}{x(x-2^{-r})} + \frac{8}{x^2} + \frac{2(q-p+1)}{x} + 1, \quad x \in (2^{-r}, 2^{-r+1}). \end{aligned}$$

□

Applying the inequality (i) of the above lemma and Theorem B, we obtain following theorem.

**Theorem 2.2.** *Let* $0 < x < 1$.

(i) *If $\{u_k\}$ is a sequence of complex numbers such that $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \le \frac{2}{x}\left[\left(1 + \frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_n|\right], \ (\forall n, m \in \mathbb{N}, m > n).$$

(ii) *If $\{u_k\}$ is a sequence of complex numbers such that $\Delta(u_k b_k) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \frac{2}{x}\left[\left(1+\frac{1}{\cos\theta}\right)|u_n| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_m|\right],\ (\forall n, m \in \mathbb{N}, m > n).$$

Specially for $b_k = 1$ we get the following inequalities for Walsh polynomials with complex monotone coefficients.

**Corollary 2.3.** *Let $0 < x < 1$. If $\{u_k\}$ is a sequence of complex numbers such that $\Delta u_k \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k \omega_k(x)\right| \leq \frac{2}{x}\left[\left(1+\frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}|u_n|\right],\ (\forall n, m \in \mathbb{N}, m > n).$$

**Corollary 2.4.** *Let $0 < x < 1$. If $\{u_k\}$ is a complex monotone sequence such that $\lim_{k\to\infty} u_k = 0$, then*

$$\left|\sum_{k=n}^{\infty} u_k \omega_k(x)\right| \leq \frac{2}{x\cos\theta}|u_n|.$$

In [4] (chapter XXI), [5] Mitrinović and Pečarić obtained inequalities for cosine and sine polynomials with monotone nonnegative coefficients. Applying Theorem 2.2, we get analogical results for Walsh polynomials with monotone nonnegative coefficients.

**Corollary 2.5.** *Let $0 < x < 1$.*

(i) *If $\{a_k\}$ is a nonnegative sequence such that $\{a_k b_k^{-1}\}$ is a decreasing sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \frac{a_n}{x}\left(\frac{b_m}{b_n}\right), \quad (\forall n, m \in \mathbb{N}, m > n).$$

(ii) *If $\{a_k\}$ is a nonnegative sequence such that $\{a_k b_k\}$ is an increasing sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \frac{a_m}{x}\left(\frac{b_m}{b_n}\right), \quad (\forall n, m \in \mathbb{N}, m > n).$$

Now, applying the inequality (ii) of Lemma 2.1, we obtain new inequalities for Walsh polynomials with complex semi-convex coefficients.

**Theorem 2.6.**

(i) *If $\{u_k\}$ is a sequence of complex numbers such that $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

*for all $n, m, r \in \mathbb{N}, m > n$.*

(ii) *If* $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2(u_k b_k) \in K(\theta)$*, then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

*for all* $n, m, r \in \mathbb{N}, m > n$.

*Proof.* (i) Applying Abel's transformation twice and the triangle inequality, we get:

$$\begin{aligned} \left|\sum_{k=n}^{m} \frac{u_k}{b_k}(b_k w_k)\right| &= \left|\frac{u_m}{b_m}\sum_{k=n}^{m} b_k w_k + \Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\sum_{j=n}^{m-1}\sum_{k=n}^{j} b_k w_k \right. \\ &\qquad \left. + \sum_{r=n}^{m-2}\Delta^2\left(\frac{u_r}{b_r}\right)\sum_{j=n}^{r}\sum_{k=n}^{j} b_k w_k\right| \\ &\leq \frac{|u_m|}{b_m} b_m \left|\sum_{k=n}^{m} w_k\right| + b_{m-1}\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\left|\sum_{j=n}^{m-1}\sum_{k=n}^{j} w_k\right| \\ &\qquad + b_{m-2}\sum_{r=n}^{m-2}\left|\Delta^2\left(\frac{u_r}{b_r}\right)\right|\left|\sum_{j=n}^{r}\sum_{k=n}^{j} w_k\right|. \end{aligned}$$

Using the Petrović inequality and inequality (ii) of Lemma 2.1, we obtain:

$$\begin{aligned} \left|\sum_{k=n}^{m} u_k w_k(x)\right| &\leq |u_m|\left|\sum_{k=n}^{m} w_k\right| + b_{m-1}\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\left|\sum_{j=n}^{m-1}\sum_{k=n}^{j} w_k\right| \\ &\qquad + \frac{b_{m-2}}{\cos\theta}\left|\sum_{r=n}^{m-2}\Delta^2\left(\frac{u_r}{b_r}\right)\sum_{j=n}^{r}\sum_{k=n}^{j} w_k\right| \\ &\leq \begin{cases} C_1(m,n,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases} \end{aligned}$$

(ii) Analogously as the proof of (i), we obtain:

$$\begin{aligned}\left|\sum_{k=n}^{m}(u_kb_k)b_k^{-1}w_k\right| &= \left|u_nb_n\sum_{k=n}^{m}b_k^{-1}w_k - \sum_{j=n+1}^{m-1}\Delta^2(u_{j-1}b_{j-1})\sum_{r=n}^{j}\sum_{k=r}^{m}b_k^{-1}w_k\right.\\ &\qquad \left.+\Delta(u_nb_n)\sum_{k=n}^{m}b_k^{-1}w_k - \Delta(u_{m-1}b_{m-1})\sum_{r=n}^{m}\sum_{k=r}^{m}b_k^{-1}w_k\right|\\ &\le |u_n|b_nb_n^{-1}\left|\sum_{k=n}^{m}w_k\right| + b_n^{-1}\sum_{j=n+1}^{m-1}\left|\Delta^2(u_{j-1}b_{j-1})\right|\left|\sum_{r=n}^{j}\sum_{k=r}^{m}w_k\right|\\ &\qquad + b_n^{-1}|\Delta(u_nb_n)|\left|\sum_{k=n}^{m}w_k\right| + b_n^{-1}|\Delta(u_{m-1}b_{m-1})|\left|\sum_{r=n}^{m}\sum_{k=r}^{m}w_k\right|.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\sum_{k=n}^{m}u_kw_k(x)\right| &\le |u_n|\left|\sum_{k=n}^{m}w_k\right| + \frac{b_n^{-1}}{\cos\theta}\left|\sum_{j=n+1}^{m-1}\Delta^2(u_{j-1}b_{j-1})\sum_{r=n}^{j}\sum_{k=r}^{m}w_k\right|\\ &\qquad + b_n^{-1}|\Delta(u_nb_n)|\left|\sum_{k=n}^{m}w_k\right| + b_n^{-1}|\Delta(u_{m-1}b_{m-1})|\left|\sum_{r=n}^{m}\sum_{k=r}^{m}w_k\right|\\ &\le \begin{cases} C_1(m,n,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right.\\ \qquad \left.\times(|\Delta(u_nb_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : 0<x<1,\\ C_2(m,n,r,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right.\\ \qquad \left.\times(|\Delta(u_nb_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : x\in(2^{-r},2^{-r+1}).\end{cases}\end{aligned}$$

□

If $b_k = 1, k = n, n+1, \dots, m$ from Theorem 2.6, we obtain the following corollary.

**Corollary 2.7.** *Let $\{u_k\}$ be a complex-convex sequence. Then,*

$$\left|\sum_{k=n}^{m}u_kw_k(x)\right| \le \begin{cases} C_1(m,n,x)\left[|u_m| + \left(1+\frac{1}{\cos\theta}\right)\right.\\ \qquad \left.\times|\Delta u_{m-1}| + \frac{1}{\cos\theta}|\Delta u_n|\right] : 0<x<1\\ C_2(m,n,r,x)\left[|u_m| + \left(1+\frac{1}{\cos\theta}\right)\right.\\ \qquad \left.\times|\Delta u_{m-1}| + \frac{1}{\cos\theta}|\Delta u_n|\right] : x\in(2^{-r},2^{-r+1})\end{cases}$$

*for all* $n,m,r\in\mathbb{N}, m>n$.

**Remark 2.8.** Similarly, the results of Theorem 2.2, Theorem 2.6, Corollary 2.3,Corollary 2.5 and Corollary 2.7 were given by the author in [7, 8] for trigonometric polynomials with complex valued coefficients.

**Corollary 2.9.**

(i) *If* $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k^{-1}\}$ *is a convex sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \le \begin{cases} C_1(m,n,x)\left[|a_m| + 2b_{m-1}\left|\Delta\left(\frac{a_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad\qquad \left. +b_{m-2}\left|\Delta\left(\frac{a_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|a_m| + 2b_{m-1}\left|\Delta\left(\frac{a_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad\qquad \left. +b_{m-2}\left|\Delta\left(\frac{a_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

*for all* $n, m, r \in \mathbb{N}, m > n$.

(ii) *If* $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k\}$ *is a convex sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \le \begin{cases} C_1(m,n,x)\left[|a_n| + 2b_n^{-1}|\Delta(a_n b_n)| \right. \\ \qquad\qquad \left. +|\Delta(a_{m-1}b_{m-1})|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|a_n| + 2b_n^{-1}|\Delta(a_n b_n)| \right. \\ \qquad\qquad \left. +|\Delta(a_{m-1}b_{m-1})|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

*for all* $n, m, r \in \mathbb{N}, m > n$.

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