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INEQUALITIES FOR WALSH POLYNOMIALS WITH SEMI-MONOTONE AND SEMI-CONVEX COEFFICIENTS

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Abstract

Using the concept of majorant sequences (see [4, ch. XXI], [5], [7], [8]) some new inequalities for Walsh polynomials with complex semi-monotone, complex semi-convex, complex monotone and complex convex coefficients are given.

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References

1. Introduction and Preliminaries

We consider the Walsh orthonormal system $\{w_n(x)\}_{n=0}^{\infty}$ defined on $[0,1)$ in the Paley enumeration. Thus $w_0(x) \equiv 1$ and for each positive integer with dyadic development

$$n = \sum_{i=1}^{p} 2^{\nu_i}, \quad \nu_1 > \nu_2 > \cdots > \nu_p \geq 0,$$

we have

$$w_n(x) = \prod_{i=1}^{p} r_{\nu_i}(x),$$

where $\{r_n(x)\}_{n=0}^{\infty}$ denotes the Rademacher system of functions defined by (see, e.g. [1, p. 60], [3, p. 9-10])

$$r_\nu(x) = \operatorname{sign} \sin 2^\nu \pi(x) \quad (\nu = 0, 1, 2, \ldots;\ 0 \leq x < 1).$$

In this paper we shall consider the Walsh polynomials $\sum_{k=n}^{m} \lambda_k w_k(x)$ with complex-valued coefficients $\{\lambda_k\}$.

Let $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \Delta\lambda_n - \Delta\lambda_{n+1} = \lambda_n - 2\lambda_{n+1} + \lambda_{n+2}$, for all $n = 1, 2, 3 \ldots$.

Petrović [6] proved the following complementary triangle inequality for a sequence of complex numbers $\{z_1, z_2, \ldots, z_n\}$.

Theorem A. *Let α be a real number and $0 < \theta < \frac{\pi}{2}$. If $\{z_1, z_2, \ldots, z_n\}$ are complex numbers such that $\alpha - \theta \leq \arg z_\nu \leq \alpha + \theta$, $\nu = 1, 2, \ldots, n$, then*

$$\left| \sum_{\nu=1}^{n} z_\nu \right| \geq (\cos\theta) \sum_{\nu=1}^{n} |z_\nu|.$$

For $0 < \theta < \frac{\pi}{2}$ denote by $K(\theta)$ the cone $K(\theta) = \{z : |\arg z| \leq \theta\}$.

Let $\{b_k\}$ be a positive nondecreasing sequence. The following definitions are given in [7] and [8]. The sequence of complex numbers $\{u_k\}$ is said to be **complex semi−monotone** if there exists a cone $K(\theta)$ such that $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$ or $\Delta(u_k b_k) \in K(\theta)$. For $b_k = 1$, the sequence $\{u_k\}$ shall be called a **complex monotone** sequence. On the other hand, the sequence $\{u_k\}$ is said to be **complex semi−convex** if there exists a cone $K(\theta)$, such that $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$ or $\Delta^2(u_k b_k) \in K(\theta)$. For $b_k = 1$, the sequence $\{u_k\}$ shall be called a **complex convex sequence**.

The following two Theorems were proved by Tomovski in [7] and [8].

Theorem B ([7]). *Let* $\{z_k\}$ *be a sequence such that* $\left|\sum_{k=n}^{m} z_k\right| \leq A,\ (\forall n, m \in \mathbb{N}, m > n)$, *where* A *is a positive number.*

(i) If $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \leq A\left[\left(1 + \frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_n|\right], \quad (\forall n, m \in \mathbb{N}, m > n).$$

(ii) If $\Delta(u_k b_k) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \leq A\left[\left(1 + \frac{1}{\cos\theta}\right)|u_n| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_m|\right], \quad (\forall n, m \in \mathbb{N}, m > n).$$

Theorem C ([8]). *Let* $A = \max_{n \leq p \leq q \leq m}\left|\sum_{j=p}^{q}\sum_{k=i}^{j} z_k\right|$.

(i) If $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \leq A\left[|u_m| + b_m\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_m}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right],$$

$$(\forall n, m \in \mathbb{N}, m > n).$$

(ii) If $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2(u_k b_k) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k z_k\right| \leq A\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\left(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|\right)\right],$$

$$(\forall n, m \in \mathbb{N}, m > n).$$

Using the concept of majorant sequences we shall give some estimates for Walsh polynomials with complex semi-monotone, complex monotone, complex semi-convex and complex convex coefficients.

2. Main Results

For the main results we require the following Lemma.

Lemma 2.1. *For all* $p, q, r \in \mathbb{N}, p < q$ *the following inequalities hold:*

(i) $$\left|\sum_{k=p}^{q} \omega_k(x)\right| \leq \frac{2}{x}, 0 < x < 1.$$

(ii) $$\left|\sum_{j=p}^{q}\sum_{k=l}^{j} \omega_k(x)\right|$$
$$\leq \begin{cases} \frac{2(q-p+1)}{x} = C_1(p,q,x) : 0 < x < 1 \\ \frac{8}{x(x-2^{-r})} + \frac{8}{x^2} + \frac{2(q-p+1)}{x} + 1 = C_2(p,q,r,x) : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

Proof. (i) Let $D_q(x) = \sum_{i=0}^{q-1} w_i(x)$ be the Dirichlet kernel. Then it is known that (see [3, p. 28]) $|D_q(x)| \leq \frac{1}{x}$, $0 < x < 1$. Hence

$$\left|\sum_{k=p}^{q} w_k(x)\right| = |D_{q+1}(x) - D_p(x)| \leq |D_{q+1}(x)| + |D_p(x)| \leq \frac{2}{x}.$$

(ii) By (i) we get

$$\left|\sum_{j=p}^{q}\sum_{k=l}^{j} w_k(x)\right| \leq \sum_{j=p}^{q}\left|\sum_{k=l}^{j} w_k(x)\right| \leq \frac{2(q-p+1)}{x}, \quad 0 < x < 1.$$

Let $F_n(x) = \frac{1}{n+1}\sum_{k=0}^{n} D_k(x)$ be the Fejer kernel. Applying Fine's inequality (see [2])

$$(n+1)F_n(x) < \frac{4}{x(x-2^{-r})} + \frac{4}{x^2}, \quad x \in (2^{-r}, 2^{-r+1}),$$

we get

$$\begin{aligned}
\left|\sum_{j=p}^{q}\sum_{k=l}^{j} w_k(x)\right| &= \left|\sum_{j=p}^{q}(D_{j+1}(x) - D_l(x))\right| \\
&\le \left|\sum_{j=p}^{q} D_{j+1}(x)\right| + \frac{q-p+1}{x} \\
&\le |(q+1)F_q(x)| + |D_{q+1}(x)| + |D_0(x)| \\
&\qquad + |pF_{p-1}(x)| + \frac{q-p+1}{x} \\
&< \frac{8}{x(x-2^{-r})} + \frac{8}{x^2} + \frac{2(q-p+1)}{x} + 1, \quad x \in (2^{-r}, 2^{-r+1}).
\end{aligned}$$

□

Applying the inequality (i) of the above lemma and Theorem B, we obtain following theorem.

Theorem 2.2. *Let* $0 < x < 1$.

(i) If $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta\left(\frac{u_k}{b_k}\right) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \le \frac{2}{x}\left[\left(1 + \frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_n|\right],$$

$$(\forall n, m \in \mathbb{N}, m > n).$$

(ii) If $\{u_k\}$ is a sequence of complex numbers such that $\Delta(u_k b_k) \in K(\theta)$, then

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \le \frac{2}{x}\left[\left(1+\frac{1}{\cos\theta}\right)|u_n| + \frac{1}{\cos\theta}\frac{b_m}{b_n}|u_m|\right],$$
$$(\forall n, m \in \mathbb{N}, m > n).$$

Specially for $b_k = 1$ we get the following inequalities for Walsh polynomials with complex monotone coefficients.

Corollary 2.3. *Let $0 < x < 1$. If $\{u_k\}$ is a sequence of complex numbers such that $\Delta u_k \in K(\theta)$, then*

$$\left|\sum_{k=n}^{m} u_k \omega_k(x)\right| \le \frac{2}{x}\left[\left(1+\frac{1}{\cos\theta}\right)|u_m| + \frac{1}{\cos\theta}|u_n|\right], \quad (\forall n, m \in \mathbb{N}, m > n).$$

Corollary 2.4. *Let $0 < x < 1$. If $\{u_k\}$ is a complex monotone sequence such that $\lim_{k\to\infty} u_k = 0$, then*

$$\left|\sum_{k=n}^{\infty} u_k \omega_k(x)\right| \le \frac{2}{x\cos\theta}|u_n|.$$

In [4] (chapter XXI), [5] Mitrinović and Pečarić obtained inequalities for cosine and sine polynomials with monotone nonnegative coefficients. Applying Theorem 2.2, we get analogical results for Walsh polynomials with monotone nonnegative coefficients.

Corollary 2.5. *Let* $0 < x < 1$.

(i) If $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k^{-1}\}$ *is a decreasing sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \frac{a_n}{x}\left(\frac{b_m}{b_n}\right), \quad (\forall n, m \in \mathbb{N}, m > n).$$

(ii) If $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k\}$ *is an increasing sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \frac{a_m}{x}\left(\frac{b_m}{b_n}\right), \quad (\forall n, m \in \mathbb{N}, m > n).$$

Now, applying the inequality (ii) of Lemma 2.1, we obtain new inequalities for Walsh polynomials with complex semi-convex coefficients.

Theorem 2.6.

(i) If $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2\left(\frac{u_k}{b_k}\right) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| \right. \\ \qquad \left. + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

for all $n, m, r \in \mathbb{N}, m > n$.

(ii) If $\{u_k\}$ *is a sequence of complex numbers such that* $\Delta^2(u_k b_k) \in K(\theta)$, *then*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_n| + b_n^{-1}\left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times(|\Delta(u_n b_n)| + |\Delta(u_{m-1}b_{m-1})|)\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

for all $n, m, r \in \mathbb{N}, m > n$.

Proof. (i) Applying Abel's transformation twice and the triangle inequality, we get:

$$\begin{aligned} \left|\sum_{k=n}^{m} \frac{u_k}{b_k}(b_k w_k)\right| &= \left|\frac{u_m}{b_m}\sum_{k=n}^{m} b_k w_k + \Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\sum_{j=n}^{m-1}\sum_{k=n}^{j} b_k w_k \right. \\ &\qquad \left. + \sum_{r=n}^{m-2}\Delta^2\left(\frac{u_r}{b_r}\right)\sum_{j=n}^{r}\sum_{k=n}^{j} b_k w_k\right| \\ &\leq \frac{|u_m|}{b_m} b_m \left|\sum_{k=n}^{m} w_k\right| + b_{m-1}\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\left|\sum_{j=n}^{m-1}\sum_{k=n}^{j} w_k\right| \\ &\qquad + b_{m-2}\sum_{r=n}^{m-2}\left|\Delta^2\left(\frac{u_r}{b_r}\right)\right|\left|\sum_{j=n}^{r}\sum_{k=n}^{j} w_k\right|. \end{aligned}$$

Using the Petrović inequality and inequality (ii) of Lemma 2.1, we obtain:

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \le |u_m| \left|\sum_{k=n}^{m} w_k\right| + b_{m-1}\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right|\left|\sum_{j=n}^{m-1}\sum_{k=n}^{j} w_k\right| + \frac{b_{m-2}}{\cos\theta}\left|\sum_{r=n}^{m-2}\Delta^2\left(\frac{u_r}{b_r}\right)\sum_{j=n}^{r}\sum_{k=n}^{j} w_k\right|$$

$$\le \begin{cases} C_1(m,n,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_m| + b_{m-1}\left(1+\frac{1}{\cos\theta}\right)\left|\Delta\left(\frac{u_{m-1}}{b_{m-1}}\right)\right| + \frac{b_{m-2}}{\cos\theta}\left|\Delta\left(\frac{u_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

(ii) Analogously as the proof of (i), we obtain:

$$\left|\sum_{k=n}^{m}(u_k b_k) b_k^{-1} w_k\right| = \left| u_n b_n \sum_{k=n}^{m} b_k^{-1} w_k - \sum_{j=n+1}^{m-1}\Delta^2(u_{j-1}b_{j-1})\sum_{r=n}^{j}\sum_{k=r}^{m} b_k^{-1} w_k + \Delta(u_n b_n)\sum_{k=n}^{m} b_k^{-1} w_k - \Delta(u_{m-1}b_{m-1})\sum_{r=n}^{m}\sum_{k=r}^{m} b_k^{-1} w_k\right|$$

$$\le |u_n| b_n b_n^{-1} \left| \sum_{k=n}^{m} w_k \right| + b_n^{-1} \sum_{j=n+1}^{m-1} \left| \Delta^2 (u_{j-1} b_{j-1}) \right| \left| \sum_{r=n}^{j} \sum_{k=r}^{m} w_k \right|$$
$$+ b_n^{-1} |\Delta(u_n b_n)| \left| \sum_{k=n}^{m} w_k \right| + b_n^{-1} |\Delta(u_{m-1} b_{m-1})| \left| \sum_{r=n}^{m} \sum_{k=r}^{m} w_k \right|.$$

Hence,

$$\left| \sum_{k=n}^{m} u_k w_k(x) \right| \le |u_n| \left| \sum_{k=n}^{m} w_k \right| + \frac{b_n^{-1}}{\cos\theta} \left| \sum_{j=n+1}^{m-1} \Delta^2 (u_{j-1} b_{j-1}) \sum_{r=n}^{j} \sum_{k=r}^{m} w_k \right|$$
$$+ b_n^{-1} |\Delta(u_n b_n)| \left| \sum_{k=n}^{m} w_k \right| + b_n^{-1} |\Delta(u_{m-1} b_{m-1})| \left| \sum_{r=n}^{m} \sum_{k=r}^{m} w_k \right|$$
$$\le \begin{cases} C_1(m,n,x) \left[|u_n| + b_n^{-1} \left(1 + \frac{1}{\cos\theta}\right) \right. \\ \qquad \left. \times (|\Delta(u_n b_n)| + |\Delta(u_{m-1} b_{m-1})|) \right] : 0 < x < 1, \\ C_2(m,n,r,x) \left[|u_n| + b_n^{-1} \left(1 + \frac{1}{\cos\theta}\right) \right. \\ \qquad \left. \times (|\Delta(u_n b_n)| + |\Delta(u_{m-1} b_{m-1})|) \right] : x \in (2^{-r}, 2^{-r+1}). \end{cases}$$

□

If $b_k = 1, k = n, n+1, \ldots, m$ from Theorem 2.6, we obtain the following corollary.

Corollary 2.7. *Let $\{u_k\}$ be a complex-convex sequence. Then,*

$$\left|\sum_{k=n}^{m} u_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|u_m| + \left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times|\Delta u_{m-1}| + \frac{1}{\cos\theta}|\Delta u_n|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|u_m| + \left(1+\frac{1}{\cos\theta}\right)\right. \\ \qquad \left.\times|\Delta u_{m-1}| + \frac{1}{\cos\theta}|\Delta u_n|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

for all $n, m, r \in \mathbb{N}, m > n.$

Remark 1. *Similarly, the results of Theorem 2.2, Theorem 2.6, Corollary 2.3, Corollary 2.5 and Corollary 2.7 were given by the author in [7, 8] for trigonometric polynomials with complex valued coefficients.*

Corollary 2.8.

(i) If $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k^{-1}\}$ *is a convex sequence, then*

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|a_m| + 2b_{m-1}\left|\Delta\left(\frac{a_{m-1}}{b_{m-1}}\right)\right|\right. \\ \qquad \left.+b_{m-2}\left|\Delta\left(\frac{a_n}{b_n}\right)\right|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|a_m| + 2b_{m-1}\left|\Delta\left(\frac{a_{m-1}}{b_{m-1}}\right)\right|\right. \\ \qquad \left.+b_{m-2}\left|\Delta\left(\frac{a_n}{b_n}\right)\right|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

for all $n, m, r \in \mathbb{N}, m > n.$

(ii) If $\{a_k\}$ *is a nonnegative sequence such that* $\{a_k b_k\}$ *is a convex sequence,*

then

$$\left|\sum_{k=n}^{m} a_k w_k(x)\right| \leq \begin{cases} C_1(m,n,x)\left[|a_n| + 2b_n^{-1}|\Delta(a_n b_n)|\right. \\ \qquad\qquad \left.+|\Delta(a_{m-1}b_{m-1})|\right] : 0 < x < 1 \\ C_2(m,n,r,x)\left[|a_n| + 2b_n^{-1}|\Delta(a_n b_n)|\right. \\ \qquad\qquad \left.+|\Delta(a_{m-1}b_{m-1})|\right] : x \in (2^{-r}, 2^{-r+1}) \end{cases}$$

for all $n, m, r \in \mathbb{N}, m > n$.

References

[1] G. ALEXITS, *Convergence problems of orthogonal series*, Pergamon Press New York- Oxford-Paris, 1961.

[2] N.J. FINE, On the Walsh functions, *Trans. Amer. Math. Soc.*, **65** (1949), 372–414.

[3] B.I. GOLUBOV, A.V. EFIMOV AND V.A. SKVORCOV, *Walsh series and transformations*, Science, Moscow (1987) (Russian).

[4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, 1993.

[5] D.S. MITRINOVIĆ AND J.E. PEČARIĆ, On an inequality of G.K.Lebed, *Prilozi MANU, Od. Mat. Tehn. Nauki*, **12** (1991), 15–19.

[6] M. PETROVIĆ, Theoreme sur les integrales curvilignes, *Publ. Math. Univ. Belgrade*, **2** (1933), 45–59.

[7] Ž. TOMOVSKI, On some inequalities of Mitrinovic and Pečarić, *Prilozi MANU, Od. Mat. Tehn. Nauki*, **22** (2001), 21–28

[8] Ž. TOMOVSKI, Some new inequalities for complex trigonometric polynomials with special coefficients, *J. Inequal. in Pure and Appl. Math.*, **4**(4) (2003), Art. 78. [ONLINE `http://jipam.vu.edu.au/article.php?sid=319`].