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A MINIMUM ENERGY CONDITION OF 1-DIMENSIONAL PERIODIC SPHERE PACKING

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ABSTRACT. Let $X \subset \mathbf{R}/\mathbf{Z}$ be a non-empty finite set and $f(x)$ be a real-valued function on $[0, \frac{1}{2}]$. Let an energy of X be the average value of $f(\|x-y\|)$ for $x, y \in X$ where $\|\cdot\|$ is the Euclidean distance on $\mathbf{R}/\mathbf{Z}$. Let $X_n \subset \mathbf{R}/\mathbf{Z}$ be an equally spaced n-point set. It is shown that if f is monotone decreasing and convex, then among all n-point sets, the energy is minimized by X_n. Moreover, by giving a variant of a result of Bennett and Jameson, it is shown that if f is convex, $f'(x^{\frac{1}{2}})$ is concave and $\lim_{x\to\frac{1}{2}} f'(x) = 0$, then the energy of X_n decreases with n.

Key words and phrases: Packing, Energy, One-dimension.

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1. INTRODUCTION

In digital imaging technologies, sometimes we are required to give well-dispersed points or measure the goodness of point dispersion. For example, in digital halftoning technology, there is a method called dispersed dot halftoning that provides a binary image where many small dispersed dots represent each tone of an original continuous tone image. Therefore, we want to focus on the mathematics of well-dispersed point sets' structures or estimating functions that can measure the goodness of dispersed point sets.

From a purely mathematical point of view, "sphere packing problems," are problems which are concerned with well-dispersed points. The sphere packing problems contain the problem that asks the maximum value of the minimum distance among every two points of all n dispersed point sets in the p-dimensional unit square and ask its structure [3, 5, 6]. On the other hand, in order to measure the goodness of dispersed points, there are some problems of the type that ask for a placement of points which minimize or maximize a given energy [2, 4, 3].

In general, these two problems are hard to deal with in spaces higher than 2-dimensions. However, we expect that a solution in a 1-dimensional space will give us some useful hints for

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applications to digital imaging technologies. For this purpose, we also need to combine these two problems. The reason why the usage of the sphere packing idea itself is problematic in digital discrete spaces, is because in the spaces, placements of points are limited and it is hard to remove local distortions without using the idea of energies.

Therefore, we would like to investigate the problem "if the sphere packing placement in a 1-dimensional space minimizes some energy value among every two points". For ease of analysis, we impose the target space on symmetry, that is, we work on the periodic sphere packing case [5, pp. 25]. In fact, the periodic sphere packing placement in a 1-dimensional space is precisely calculated as an equally spaced set with periodic boundary. Hence, the energy value is precisely calculated, too. Therefore, the interest is to investigate the condition and property of the energy itself that takes the minimum value when given points are equally spaced points.

For the above reasons, we investigate the following two points.

(A). A global minimum condition of the energy of equally spaced n-point sets.

(B). A condition so that the energy of equally spaced n-point sets decreases with n.

2. DEFINITION AND PRELIMINARIES

Definition 2.1. Let $E = [0, 1)$. Let the distance $\|x - y\|$ between two points $x, y \in E$ be

$$\|x - y\| = \min\{|x - y + e| : e = -1, 0, 1\}.$$

Let $X \subset (E, \|\cdot\|)$ be a non-empty finite set. Let f be any real-valued function on $\left[0, \frac{1}{2}\right]$. Then, let the energy of a point $x \in X$ be

$$I(X, x, f) = \frac{1}{|X|} \sum_{y \in X} f(\|x - y\|)$$

and the energy of the set X be

$$I(X, f) = \frac{1}{|X|} \sum_{x \in X} I(X, x, f) = \frac{1}{|X|^2} \sum_{x \in X} \sum_{y \in X} f(\|x - y\|),$$

where $|X|$ means the cardinality of X. Let $X_n \subset (E, \|\cdot\|)$ denote an equally spaced n-point set.

The space $(E, \|\cdot\|)$ is metrically equivalent to the circle S^1 with the arcwise distance, and locally equivalent to $\mathbf{R}$. By the definition of $(E, \|\cdot\|)$, $\|x - y\| \leq \frac{1}{2}$ holds for any $x, y \in (E, \|\cdot\|)$.

Remark 2.1. To compare energies for fixed f, we can assume $f\left(\frac{1}{2}\right) = 0$, because energies are written by $I(X, x, f) = I(X, x, g) + f\left(\frac{1}{2}\right)$ and $I(X, f) = I(X, g) + f\left(\frac{1}{2}\right)$ where $g(x) = f(x) - f\left(\frac{1}{2}\right)$ with $g\left(\frac{1}{2}\right) = 0$. Moreover, we can extend f onto $\left(\frac{1}{2}, \infty\right]$ by $f(x) = f\left(\frac{1}{2}\right) = 0$. Then,

$$I(X_n, x, f) = \frac{1}{n} f(0) + \frac{2}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right), \tag{2.1}$$

$$I(X_n, f) = I(X_n, x, f) \tag{2.2}$$

hold for any $X_n \subset (E, \|\cdot\|)$ and $x \in X_n$.

3. GLOBAL MINIMUM CONDITION FOR FIXED n

We investigate the problem (A) and give a sufficient condition so that the minimum energy among n-point sets is given by $I(X_n, f)$. In a 1-dimensional space, we do not need a local minimum condition analysis, because we can obtain a global minimum condition from the following.

Theorem 3.1. *Let $Y \subset (E, \|\cdot\|)$ be any n-point set. If f is monotone decreasing and convex, then $I(Y, f) \geq I(X_n, f)$ holds.*

Proof. By Remark 2.1, we can assume $f(x) = 0$ $\left(x \geq \frac{1}{2}\right)$. Then, by the assumption, f is convex on $[0, \infty)$. Let $Y = \{y_{n+1}, \dots, y_{2n}\} \subset [0, 1)$ for convenience and assume

$$y_i \leq y_{i+1} \quad (i = n+1, \dots, 2n-1).$$

Moreover, take points on both sides of the set Y as

$$\begin{aligned} y_i &= y_{i+n} - 1 \quad (i = 1, \dots, n) \\ y_i &= y_{i-n} + 1 \quad (i = 2n+1, \dots, 3n). \end{aligned}$$

Then, for $\{y_1, \dots, y_{3n}\} \subset [-1, 2)$,

$$y_i \leq y_{i+1} \quad (i = 1, \dots, 3n-1) \tag{3.1}$$

holds. Next, let

$$d_i = y_{i+1} - y_i \quad (i = 1, \dots, 3n-1). \tag{3.2}$$

Here, $d_i = d_{i+n} = d_{i+2n}$ holds for each $i = 1, \dots, n$.

By (3.1) and (3.2), for each $k = n+1, \dots, 2n$ and $i = 1, \dots, n$,

$$|\, y_k - y_{k+i} \,| = \sum_{j=1}^{i} d_{k+j-1}, \tag{3.3}$$

$$|\, y_k - y_{k-i} \,| = \sum_{j=1}^{i} d_{k-j} \tag{3.4}$$

holds. In addition, the following hold for each $i = 1, \dots, 2n$.

$$\sum_{j=1}^{n} d_{i+j} = 1. \tag{3.5}$$

By (3.3), (3.4) and $f(x) = 0$ on $x \in \left[\frac{1}{2}, \infty\right)$, the following holds for each $k = n+1, \dots, 2n$:

$$\begin{aligned} I(Y, y_k, f) &= \frac{1}{n} \sum_{y \in Y} f\left(\|y_k - y\|\right) \\ &= \frac{1}{n} \sum_{i=1}^{3n} f\left(|y_k - y_i|\right) \\ &= \frac{1}{n} \sum_{i=1}^{n} f\left(|y_k - y_{k+i}|\right) + \frac{1}{n} \sum_{i=1}^{n} f\left(|y_k - y_{k-i}|\right) + \frac{1}{n} f(0) \\ &= \frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=1}^{i} d_{k+j-1}\right) + \frac{1}{n} \sum_{i=1}^{n} f\left(\sum_{j=1}^{i} d_{k-j}\right) + \frac{1}{n} f(0). \end{aligned} \tag{3.6}$$

Since f is convex on $[0,\infty)$, by Jensen's inequality and by (3.5), (3.6), (2.1) and (2.2), the following holds.

$$\begin{aligned}
I(Y,f) &= \frac{1}{n}\sum_{y\in Y} I(Y,y,f) \\
&= \frac{1}{n}\sum_{k=n+1}^{2n} I(Y,y_k,f) \\
&= \frac{1}{n^2}\sum_{k=n+1}^{2n}\left[\sum_{i=1}^{n} f\left(\sum_{j=1}^{i} d_{k+j-1}\right) + \sum_{i=1}^{n} f\left(\sum_{j=1}^{i} d_{k-j}\right) + f(0)\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=n+1}^{2n} f\left(\sum_{j=1}^{i} d_{k+j-1}\right) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{k=n+1}^{2n} f\left(\sum_{j=1}^{i} d_{k-j}\right) + \frac{1}{n}f(0) \\
&\geq \frac{1}{n^2}\sum_{i=1}^{n}\left[nf\left(\frac{1}{n}\sum_{j=1}^{i}\sum_{k=n+1}^{2n} d_{k+j-1}\right)\right] + \frac{1}{n^2}\sum_{i=1}^{n}\left[nf\left(\frac{1}{n}\sum_{j=1}^{i}\sum_{k=n+1}^{2n} d_{k-j}\right)\right] + \frac{1}{n}f(0) \\
&= \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) + \frac{1}{n}f(0) \\
&= I(X_n,f).
\end{aligned}$$

$I(Y,f) > I(X_n,f)$ holds when f is bounded and strictly convex on $\left[0,\frac{1}{2}\right]$. This concludes the proof of Theorem 3.1. □

4. MONOTONE DECREASING CONDITION WITH n

We investigate Problem (B) and give a sufficient condition so that the energy $I(X_n,f)$ decreases with n. For this, we give a variant of a result of Bennett and Jameson [1, Theorem 5]. Let $T_n(f)$ be the trapezium estimate for $\int_0^1 f$ given by dividing $[0,1]$ into n equal subintervals:

$$T_n(f) = \frac{1}{2n}\sum_{i=0}^{n-1}\left[f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right)\right].$$

The result of [1] is that if f is convex and f' is either convex or concave, then $T_n(f)$ decreases with n. Here we show the same holds if f is convex, $f'\left(x^{\frac{1}{2}}\right)$ is concave and $\lim_{x\to 1} f'(x) = 0$.

Theorem 4.1. *Let f be a differentiable function on $[0,1]$. If $f(x)$ is convex, $f'\left(x^{\frac{1}{2}}\right)$ is concave and $\lim\limits_{x\to 1} f'(x) = 0$, then $T_n(f)$ decreases with n.*

Proof. Without loss of generality, we can assume $f(1) = 0$. Extend f onto $(1,\infty)$ by $f(x) = f(1) = 0$. Then, f is differentiable convex on $[0,\infty)$. For a real number $a \geq 1$, extend $T_n(f)$ by

$$T_a(f) = \frac{1}{2a}\sum_{i=0}^{[a]}\left[f\left(\frac{i}{a}\right) + f\left(\frac{i+1}{a}\right)\right].$$

Here, if a is a natural number, then $[a]$ can be replaced by $a-1$. We show that $T_a(f)$ decreases with $a \geq 1$ by analyzing the differential coefficient of $T_a(f)$ for a. The differentiability of $T_a(f)$ for a is guaranteed by the fact that f is differentiable on $[0,\infty)$ and $f(x) = f'(x) = 0$ for

$x \geq 1$. In fact, the differential coefficient ${T_a}'(f)$ is expressed by ${T_a}'(f) = -\frac{1}{a^2} \cdot S_a(f)$ where $S_a(f)$ is given by the following.

$$S_a(f) = \sum_{i=0}^{[a]} \left[f\left(\frac{i}{a}\right) + \frac{i}{a} f'\left(\frac{i}{a}\right) + f\left(\frac{i+1}{a}\right) + \frac{i+1}{a} f'\left(\frac{i+1}{a}\right) \right].$$

Thus, it is enough to show that $S_a(f) \geq 0$ for $a \geq 1$.

In the following, we show $S_n(f) \geq 0$ for any natural number n. Let $g(x) = x^{\frac{1}{2}}$. Here, $f' \circ g$ is concave on $[0, \infty)$, because $f' \circ g$ is concave on $[0, 1]$ and $f'(x) = 0\,(x \geq 1)$. Hence, we get the following inequality for i by considering the trapezium estimate for the integral of $f' \circ g$ (Figure 4.1).

$$\begin{aligned} \int_{\frac{i^2-i}{n^2}}^{\frac{i^2+i}{n^2}} f' \circ g(x)dx &\leq \frac{2i}{n^2} f' \circ g\left(\left(\frac{i}{n}\right)^2\right) \\ &= \frac{2i}{n^2} f'\left(\frac{i}{n}\right). \end{aligned}$$

Thus, by remarking $f(x) = f'(x) = 0$ on $x \geq 1$, we get the following.

$$\begin{aligned} \sum_{i=0}^{n} \frac{2i}{n^2} f'\left(\frac{i}{n}\right) &\geq \sum_{i=0}^{n} \int_{\frac{i^2-i}{n^2}}^{\frac{i^2+i}{n^2}} f' \circ g(x)dx \\ &= \int_0^1 f' \circ g(x)dx \\ &= 2\left[xf(x)\right]_0^1 - 2\int_0^1 f(x)dx \\ &= -2\int_0^1 f(x)dx. \end{aligned}$$

Therefore, by remarking that $f'(x) = 0$ on $x \geq 1$ again, we obtain:

$$\begin{aligned} S_n(f) &= \sum_{i=0}^{n-1} \left[f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right) + \frac{i}{n} f'\left(\frac{i}{n}\right) + \frac{i+1}{n} f'\left(\frac{i+1}{n}\right) \right] \\ &= \sum_{i=0}^{n-1} \left[f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right) + \frac{2}{n} i f'\left(\frac{i}{n}\right) \right] \\ &= \sum_{i=0}^{n-1} \left[f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right) \right] + \sum_{i=0}^{n} \frac{2}{n} i f'\left(\frac{i}{n}\right) \\ &\geq \sum_{i=0}^{n-1} \left[f\left(\frac{i}{n}\right) + f\left(\frac{i+1}{n}\right) \right] - 2n\int_0^1 f(x)dx \\ &\geq 0. \end{aligned}$$

The last inequality holds by the the trapezium estimate for the integral of f which is convex on $[0, 1]$. Thus, $S_n(f) \geq 0$ holds for any natural number n.

We note again the condition that f is convex on $[0, \infty)$, $f'\big(x^{\frac{1}{2}}\big)$ is concave on $[0, \infty)$ and $f(x) = f'(x) = 0$ for $x \geq 1$. Then, we get $S_a(f) \geq 0$ for any real number $a \geq 1$ in the same manner as for natural number n.

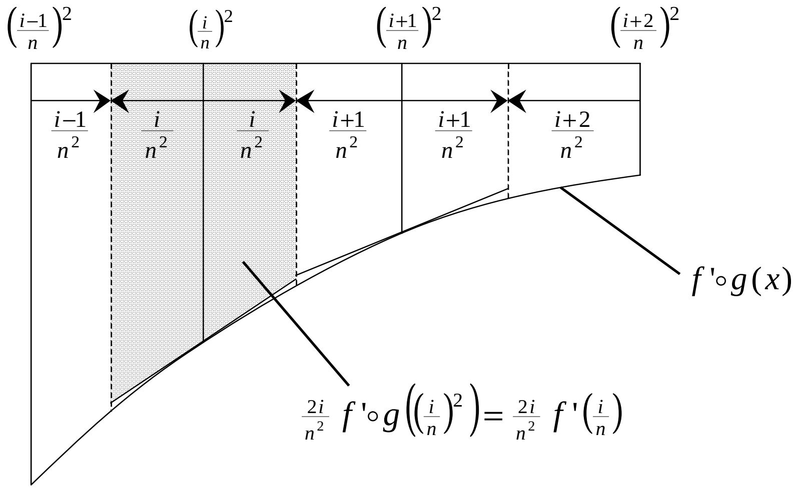

Figure 4.1: A trapezium estimate for the integral of the concave function $f' \circ g$ where $g(x) = x^{\frac{1}{2}}$.

Therefore, $T_n(f)$ decreases with n. $T_n(f)$ strictly decreases with n when f is bounded and strictly convex on $[0, 1]$. By the assumption of this theorem, f is indeed a monotone decreasing C^1 class function. This concludes the proof of Theorem 4.1. $\square$

Now we apply Theorem 4.1 to the energy.

Corollary 4.2. *If $f(x)$ is differentiable convex, $f'\left(x^{\frac{1}{2}}\right)$ is concave and $\lim_{x\to\frac{1}{2}} f'(x) = 0$, then $I(X_n, f)$ decreases with n.*

Proof. By Remark 2.1, we can assume $f(x) = 0 \ \left(x \geq \frac{1}{2}\right)$ and then $I(X_n, f) = 2T_n(f)$ holds by (2.1) and (2.2). By the assumption, f is differentiable on $[0, 1]$ with $\lim_{x\to 1} f'(x) = 0$. Hence, by Theorem 4.1, $I(X_n, f)$ decreases with n. $\square$

Although Theorem 3.1 guarantees that the energy is minimized by equally spaced points, it does not necessarily guarantee that the energy takes lower value by more dispersed points. Corollary 4.2 guarantees the latter matter and reinforces Theorem 3.1.

5. SUMMARY AND EXAMPLES

We summarize the previous results in the following Corollary 5.1 by changing f to a real-valued function h defined on $[0, 1]$. In potential theory, potential energies are determined as $\int\int |x-y|^{-s} d\mu(x) d\mu(y)$ for real number s [2]. However, it is convenient to give the sphere of influence r $(0 < r \leq \frac{1}{2})$ for each point, for our purpose of application to digital imaging technologies, for ease of calculations. That is, points that are placed in a distance larger than r from a noticed point do not influence the energy.

Corollary 5.1. *Let r be $0 < r \leq \frac{1}{2}$. For the energy defined by Definition 2.1, let*

$$f(|x-y|) = \begin{cases} h\left(\dfrac{|x-y|}{r}\right), & |x-y| < r \\ 0, & |x-y| \geq r \end{cases}$$

where h is a real-valued function on $[0,1]$. Assume that the following conditions are satisfied.
(H1) $h(x)$ *is monotone decreasing.*
(H2) $h(x)$ *is differentiable on* $[0,1]$.
(H3) $h(1)=0$.
(H4) $\lim_{x\to 1} h'(x)=0$.
(H5) $h(x)$ *is convex.*
(H6) $h'(x^{\frac{1}{2}})$ *is concave.*
Then, for any m-point set $Y\subset(E,\|\cdot\|)$ with $1\le m\le n$, $I(Y,f)\ge I(X_n,f)$ holds.

Proof. From H2, H3 and H4, f is differentiable. From H1 and H5, $f(x)$ is monotone decreasing and convex. From H4 and H6, $f'\left(x^{\frac{1}{2}}\right)$ is concave on $\left[0,\frac{1}{2}\right]$ and $\lim_{x\to\frac{1}{2}} f'(x)=0$. Hence, by Theorem 3.1 and Corollary 4.2, $I(Y,f)\ge I(X_n,f)$ holds. $I(Y,f)>I(X_n,f)$ holds when h is bounded, strictly convex on $[0,1]$ and $r\ge\frac{1}{n}$. In fact, the condition H1 is derived from H4 and H5. From H2 and H6, h is indeed a C^1 class function. □

We give the following examples for the function h of the Corollary 5.1.

Let $p>0$. In the following, we consider four examples as the function h.

Example 5.1. $h_1(x)=(1-x)^p$.
$h_1(x)$ is monotone decreasing and convex if and only if $p\ge 1$. Since ${h_1}'\left(x^{\frac{1}{2}}\right)=-p(1-x^{\frac{1}{2}})^{p-1}$, H1-H6 all hold for $p\ge 2$. In the case of $p=1$, H4 and H6 do not hold. In the case of $1<p<2$, H6 does not hold.

Example 5.2. $h_2(x)=\left(\frac{2}{3}-x+\frac{1}{3}x^3\right)^p$.
$h_2(x)$ is monotone decreasing and convex if and only if $p\ge 1$. Since ${h_2}'(x)=p\left(\frac{2}{3}-x+\frac{1}{3}x^3\right)^{p-1}\cdot(x^2-1)$, ${h_2}'\left(x^{\frac{1}{2}}\right)$ is concave when $p\ge 1$, too. Thus, in the case of $p\ge 1$, H1-H6 all hold. This function gives the power of p value of the intersection volume of two 3-dimensional balls with diameter 1, where the center points of these balls are placed with distance x (Figure 5.1(upper)).

Example 5.3. $h_3(x)=1-x+x(x^p-1)/p$.
$h_3(x)$ is monotone decreasing and convex for all $p>0$. In the case of $p=1$, $h_3(x)$ equals to the function $h_1(x)$ with $p=2$, and in the case of $p=2$, $h_3(x)$ equals to the function $\frac{3}{2}h_2(x)$ with $p=1$. As p goes to infinity, $h_3(x)$ converges to the function $1-x$. Since ${h_3}'(x)=-1+(p+1)/px^p-1/p$, ${h_3}'\left(x^{\frac{1}{2}}\right)$ is concave when $0<p\le 2$ and H1-H6 all hold. In the case of $p>2$, H6 does not hold.

Example 5.4. $h_4(x)=\left(2\cos^{-1}(x)-\sin(2\cos^{-1}(x))\right)^p$.
$h_4(x)$ is monotone decreasing and convex when $p\ge 1$. Since

$${h_4}'(x)=p\left(2\cos^{-1}(x)-\sin(2\cos^{-1}(x))\right)^{p-1}\cdot\frac{-2}{\sqrt{1-x^2}}\cdot(1-\cos(2\cos^{-1}(x))),$$

one can check that ${h_4}'\left(x^{\frac{1}{2}}\right)$ is concave if p is to some extent large, for example $p>1.5$. At least, in the case of $p=1$, ${h_4}'\left(x^{\frac{1}{2}}\right)$ is not concave. This function gives the power of p value of the intersection area of two circles with diameter 1, where the center points of these circles are placed with distance x (Figure 5.1(lower)). Therefore, although the function h_2 with $p=1$, which is the volume of the cross region of 2 balls, satisfies the condition H1-H6, the function h_4 with $p=1$, which is the area of the cross region of 2 circles, does not satisfy the condition H6.

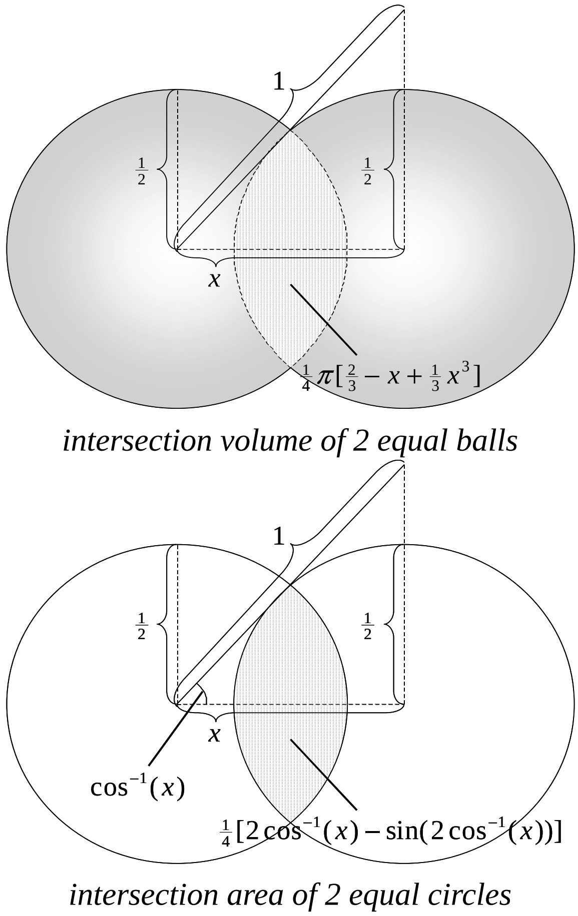

Figure 5.1: Illustrations of functions $h_2(x)$ (upper) and $h_4(x)$ (lower).

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