

TWO NEW MAPPINGS ASSOCIATED WITH INEQUALITIES OF HADAMARD-TYPE FOR CONVEX FUNCTIONS

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Abstract: In this paper, we define two mappings associated with the Hadamard inequality, investigate their main properties and give some refinements.



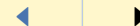
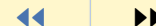
Mappings Associated with
Inequalities of Hadamard-type

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vol. 10, iss. 3, art. 81, 2009

[Title Page](#)

[Contents](#)



Page 1 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

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Contents

1	Introduction	3
2	Main Results	5
3	Proof of Theorems	7



**Mappings Associated with
Inequalities of Hadamard-type**

Lan He

vol. 10, iss. 3, art. 81, 2009

[Title Page](#)

[Contents](#)



Page 2 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

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mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 3 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

1. Introduction

Let $f, -g : [a, b] \rightarrow \mathbb{R}$ both be continuous functions. If f is a convex function, then we have

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt.$$

The inequality (1.1) is well known as the Hadamard inequality (see [1] – [6]). For some recent results which generalize, improve, and extend this classical inequality, see the references of [3].

When $f, -g$ both are convex functions satisfying $\int_a^b g(x)dx > 0$ and $f\left(\frac{a+b}{2}\right) \geq 0$, S.-J. Yang in [7] generalized (1.1) as

$$(1.2) \quad \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}.$$

To go further in exploring (1.2), we define two mappings L and F by $L : [a, b] \times [a, b] \mapsto \mathbb{R}$,

$$L(x, y; f, g) = \left[\int_x^y f(t)dt - (y-x)f\left(\frac{x+y}{2}\right) \right] \left[(y-x)g\left(\frac{x+y}{2}\right) - \int_x^y g(t)dt \right]$$

and $F : [a, b] \times [a, b] \mapsto \mathbb{R}$,

$$F(x, y; f, g) = g\left(\frac{x+y}{2}\right) \int_x^y f(t)dt - f\left(\frac{x+y}{2}\right) \int_x^y g(t)dt.$$

The aim of this paper is to study the properties of L and F and obtain some new refinements of (1.2).

To prove the theorems of this paper we need the following lemma.



Title Page

Contents



Page 4 of 11

Go Back

Full Screen

Close

Lemma 1.1. Let f be a convex function on $[a, b]$. The mapping H is defined as

$$H(x, y; f) = \int_x^y f(t)dt - (y - x)f\left(\frac{x + y}{2}\right).$$

Then $H(a, y; f)$ is nonnegative and monotonically increasing with y on $[a, b]$ (see [8]), $H(x, b; f)$ is nonnegative and monotonically decreasing with x on $[a, b]$ (see [9]).



[Title Page](#)

[Contents](#)

◀◀ ▶▶

◀ ▶

Page 5 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

2. Main Results

The properties of L are embodied in the following theorem.

Theorem 2.1. *Let f and $-g$ both be convex functions on $[a, b]$. Then we have:*

1. $L(a, y; f, g)$ is nonnegative increasing with y on $[a, b]$, $L(x, b; f, g)$ is nonnegative decreasing with x on $[a, b]$.
2. When $\int_a^b g(x)dx > 0$ and $f\left(\frac{a+b}{2}\right) \geq 0$, for any $x, y \in (a, b)$ and $\alpha \geq 0$ and $\beta \geq 0$ such that $\alpha + \beta = 1$, we have the following refinement of (1.2)

$$\begin{aligned}
 (2.1) \quad \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} &\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)dt} + \frac{\int_a^b f(t)dt}{2(b-a)g\left(\frac{a+b}{2}\right)} \\
 &\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)dt} + \frac{\int_a^b f(t)dt}{2(b-a)g\left(\frac{a+b}{2}\right)} \\
 &\quad + \frac{\alpha L(a, y; f, g) + \beta L(x, b; f, g)}{2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt} \\
 &\leq \frac{\int_a^b f(t)dt}{2\int_a^b g(t)dt} + \frac{2f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}.
 \end{aligned}$$

The main properties of F are given in the following theorem.

Theorem 2.2. *Let f and $-g$ both be nonnegative convex functions on $[a, b]$ satisfying $\int_a^b g(x)dx > 0$. Then we have the following two results:*



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 11

Go Back

Full Screen

Close

1. If f and $-g$ both are increasing, then $F(a, y; f, g)$ is nonnegative increasing with y on $[a, b]$, and we have the following refinement of (1.2)

$$(2.2) \quad \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(a, y; f, g)}{g\left(\frac{a+b}{2}\right) \int_a^b g(t) dt} \leq \frac{\int_a^b f(t) dt}{\int_a^b g(t) dt},$$

where $y \in (a, b)$.

2. If f and $-g$ both are decreasing, then $F(x, b; f, g)$ is nonnegative decreasing with x on $[a, b]$, and we have the following refinement of (1.2)

$$(2.3) \quad \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(x, b; f, g)}{g\left(\frac{a+b}{2}\right) \int_a^b g(t) dt} \leq \frac{\int_a^b f(t) dt}{\int_a^b g(t) dt},$$

where $x \in (a, b)$.

3. Proof of Theorems

Proof of Theorem 2.1.

(1) By Lemma 1.1 and the convexity of f and $-g$, it is obvious that $H(a, y; f)$ and $H(a, y; -g)$ both are nonnegative increasing with y on $[a, b]$. Then $L(a, y; f, g) = H(a, y; f)H(a, y; -g)$ is nonnegative increasing with y on $[a, b]$. By the same arguments of proof for $L(a, y; f, g)$, we can also prove that $L(x, b; f, g)$ is nonnegative decreasing with x on $[a, b]$.

(2) Since $H(a, y; f)$ is monotonically increasing with y on $[a, b]$, for any $y \in (a, b)$ and $\alpha \geq 0$, we have

$$(3.1) \quad 0 = \alpha L(a, a; f, g) \leq \alpha L(a, y; f, g) \leq \alpha L(a, b; f, g).$$

As $H(x, b; f)$ is monotonically decreasing with x on $[a, b]$, for any $x \in (a, b)$ and $\beta \geq 0$, we have

$$(3.2) \quad 0 = \beta L(a, a; f, g) \leq \beta L(x, b; f, g) \leq \beta L(a, b; f, g).$$

When $\alpha + \beta = 1$, expression (3.1) plus (3.2) yields

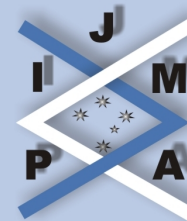
$$(3.3) \quad 0 = L(a, a; f, g) \leq \alpha L(a, y; f, g) + \beta L(x, b; f, g) \leq L(a, b; f, g).$$

Expression (3.3) plus

$$(b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt$$

yields

$$(3.4) \quad (b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt$$



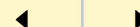
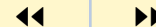
Mappings Associated with
Inequalities of Hadamard-type

Lan He

vol. 10, iss. 3, art. 81, 2009

Title Page

Contents



Page 7 of 11

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

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Title Page

Contents



Page 8 of 11

Go Back

Full Screen

Close

$$\begin{aligned} &\leq (b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt \\ &\quad + \alpha L(a, y; f, g) + \beta L(x, b; f, g) \\ &\leq (b-a)g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt + (b-a)f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt. \end{aligned}$$

By the convexity of f and g , $\int_a^b g(x)dx > 0$, $f\left(\frac{a+b}{2}\right) \geq 0$ and (1.1), we get

$$(3.5) \quad (b-a)g\left(\frac{a+b}{2}\right) \geq \int_a^b g(t)dt > 0, \quad \int_a^b f(t)dt \geq (b-a)f\left(\frac{a+b}{2}\right) \geq 0.$$

Using (3.5), we obtain

$$\begin{aligned} (3.6) \quad &(b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt \\ &\geq (b-a)f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt + (b-a)f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt \\ &= 2(b-a)f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt \end{aligned}$$

and

$$\begin{aligned} (3.7) \quad &(b-a)g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt + (b-a)f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt \\ &\leq 2(b-a)g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt. \end{aligned}$$



Title Page

Contents



Page 9 of 11

Go Back

Full Screen

Close

Combining (3.4), (3.6) and (3.7), and dividing the combined formula by

$$2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$

yields (2.1).

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2.

(1) By Lemma 1.1 and the convexity of f and $-g$, we can see that $H(a, y; f)$ and $H(a, y; -g)$ both are nonnegative increasing with y on $[a, b]$. From the nonnegative increasing properties of f and g , we get that

$$\begin{aligned} F(a, y; f, g) &= g\left(\frac{a+y}{2}\right)\int_a^y f(t)dt - f\left(\frac{a+y}{2}\right)\int_a^y g(t)dt \\ &= g\left(\frac{a+y}{2}\right)\left(\int_a^y f(t)dt - (y-a)f\left(\frac{a+y}{2}\right)\right) \\ &\quad + f\left(\frac{a+y}{2}\right)\left(\int_a^y g(t)dt - (y-a)g\left(\frac{a+y}{2}\right)\right) \\ &= g\left(\frac{a+y}{2}\right) \cdot H(a, y; f) + f\left(\frac{a+y}{2}\right) \cdot H(a, y; -g) \end{aligned}$$

is nonnegative increasing with y on $[a, b]$.

Since $F(a, y; f, g)$ is monotonically increasing with y on $[a, b]$, for any $y \in (a, b)$, we have

$$(3.8) \quad 0 = F(a, a; f, g) \leq F(a, y; f, g) \leq F(a, b; f, g).$$

Expression (3.8) plus

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 11

Go Back

Full Screen

Close

yields

$$\begin{aligned}(3.9) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt &\leq f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt + F(a, y; f, g) \\ &\leq f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt + F(a, b; f, g) \\ &= g\left(\frac{a+b}{2}\right) \int_a^b f(t) dt.\end{aligned}$$

Expression (3.9) divided by

$$g\left(\frac{a+b}{2}\right) \int_a^b g(t) dt$$

yields (2.2).

(2) By Lemma 1.1 and the convexity of f and $-g$, we can see that $H(x, b; f)$ and $H(x, b; -g)$ are both nonnegative decreasing with x on $[a, b]$. Further, from the nonnegative decreasing properties of f and g , we obtain that

$$F(x, b; f, g) = g\left(\frac{x+b}{2}\right) \cdot H(x, b; f) + f\left(\frac{x+b}{2}\right) \cdot H(x, b; -g)$$

is nonnegative decreasing with x on $[a, b]$.

For any $x \in (a, b)$, then

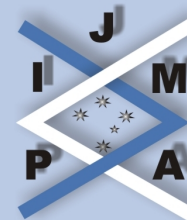
$$(3.10) \quad 0 = F(a, a; f, g) \leq F(x, b; f, g) \leq F(a, b; f, g).$$

Using (3.10), by the same arguments of proof for (1) of Theorem 2.2, we can also prove that (2.3) is true.

This completes the proof of Theorem 2.2. □

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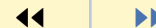
Mappings Associated with
Inequalities of Hadamard-type

Lan He

vol. 10, iss. 3, art. 81, 2009

Title Page

Contents



Page 11 of 11

Go Back

Full Screen

Close

journal of **inequalities**
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mathematics

issn: 1443-5756