TWO NEW MAPPINGS ASSOCIATED WITH INEQUALITIES OF HADAMARD-TYPE FOR CONVEX FUNCTIONS

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Abstract: In this paper, we define two mappings associated with the Hadamard inequality, investigate their main properties and give some refinements.

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1. Introduction

Let $f, -g : [a, b] \to \mathbb{R}$ both be continuous functions. If f is a convex function, then we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt. \tag{1.1}$$

The inequality (1.1) is well known as the Hadamard inequality (see [1] – [6]). For some recent results which generalize, improve, and extend this classical inequality, see the references of [3].

When $f, -g$ both are convex functions satisfying $\int_a^b g(x)dx > 0$ and $f(\frac{a+b}{2}) \geq 0$, S.-J. Yang in [7] generalized (1.1) as

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}. \tag{1.2}$$

To go further in exploring (1.2), we define two mappings L and F by $L : [a, b] \times [a, b] \mapsto \mathbb{R}$,

$$L(x, y; f, g) = \left[\int_x^y f(t)dt - (y-x)f\left(\frac{x+y}{2}\right)\right]\left[(y-x)g\left(\frac{x+y}{2}\right) - \int_x^y g(t)dt\right]$$

and $F : [a, b] \times [a, b] \mapsto \mathbb{R}$,

$$F(x, y; f, g) = g\left(\frac{x+y}{2}\right)\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right)\int_x^y g(t)dt.$$

The aim of this paper is to study the properties of L and F and obtain some new refinements of (1.2).

To prove the theorems of this paper we need the following lemma.

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Lemma 1.1. *Let f be a convex function on $[a,b]$. The mapping H is defined as*

$$H(x,y;f) = \int_x^y f(t)dt - (y-x)f\left(\frac{x+y}{2}\right).$$

Then $H(a,y;f)$ is nonnegative and monotonically increasing with y on $[a,b]$ (see [8]), $H(x,b;f)$ is nonnegative and monotonically decreasing with x on $[a,b]$ (see [9]).

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2. Main Results

The properties of L are embodied in the following theorem.

Theorem 2.1. *Let f and $-g$ both be convex functions on $[a, b]$. Then we have:*

1. $L(a, y; f, g)$ *is nonnegative increasing with* y *on* $[a, b]$, $L(x, b; f, g)$ *is nonnegative decreasing with* x *on* $[a, b]$.
2. *When* $\int_a^b g(x)dx > 0$ *and* $f\left(\frac{a+b}{2}\right) \geq 0$, *for any* $x, y \in (a, b)$ *and* $\alpha \geq 0$ *and* $\beta \geq 0$ *such that* $\alpha + \beta = 1$, *we have the following refinement of (1.2)*

$$\begin{aligned}
(2.1)\qquad \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} &\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)dt} + \frac{\int_a^b f(t)dt}{2(b-a)g\left(\frac{a+b}{2}\right)} \\
&\leq \frac{(b-a)f\left(\frac{a+b}{2}\right)}{2\int_a^b g(t)dt} + \frac{\int_a^b f(t)dt}{2(b-a)g\left(\frac{a+b}{2}\right)} \\
&\qquad + \frac{\alpha L(a, y; f, g) + \beta L(x, b; f, g)}{2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt} \\
&\leq \frac{\int_a^b f(t)dt}{2\int_a^b g(t)dt} + \frac{2f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)} \leq \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}.
\end{aligned}$$

The main properties of F are given in the following theorem.

Theorem 2.2. *Let f and $-g$ both be nonnegative convex functions on $[a, b]$ satisfying $\int_a^b g(x)dx > 0$. Then we have the following two results:*

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1. *If f and $-g$ both are increasing, then $F(a, y; f, g)$ is nonnegative increasing with y on $[a, b]$, and we have the following refinement of (1.2)*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(a, y; f, g)}{g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt} \le \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}, \tag{2.2}$$

where $y \in (a, b)$.

2. *If f and $-g$ both are decreasing, then $F(x, b; f, g)$ is nonnegative decreasing with x on $[a, b]$, and we have the following refinement of (1.2)*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} + \frac{F(x, b; f, g)}{g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt} \le \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}, \tag{2.3}$$

where $x \in (a, b)$.

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3. Proof of Theorems

Proof of Theorem 2.1.
(1) By Lemma 1.1 and the convexity of f and $-g$, it is obvious that $H(a,y;f)$ and $H(a,y;-g)$ both are nonnegative increasing with y on $[a,b]$. Then $L(a,y;f,g) = H(a,y;f)H(a,y;-g)$ is nonnegative increasing with y on $[a,b]$. By the same arguments of proof for $L(a,y;f,g)$, we can also prove that $L(x,b;f,g)$ is nonnegative decreasing with x on $[a,b]$.

(2) Since $H(a,y;f)$ is monotonically increasing with y on $[a,b]$, for any $y \in (a,b)$ and $\alpha \geq 0$, we have

$$0 = \alpha L(a,a;f,g) \leq \alpha L(a,y;f,g) \leq \alpha L(a,b;f,g). \tag{3.1}$$

As $H(x,b;f)$ is monotonically decreasing with x on $[a,b]$, for any $x \in (a,b)$ and $\beta \geq 0$, we have

$$0 = \beta L(a,a;f,g) \leq \beta L(x,b;f,g) \leq \beta L(a,b;f,g). \tag{3.2}$$

When $\alpha + \beta = 1$, expression (3.1) plus (3.2) yields

$$0 = L(a,a;f,g) \leq \alpha L(a,y;f,g) + \beta L(x,b;f,g) \leq L(a,b;f,g). \tag{3.3}$$

Expression (3.3) plus

$$(b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt$$

yields

$$(b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt \tag{3.4}$$

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$$\leq (b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt + \alpha L(a,y;f,g) + \beta L(x,b;f,g)$$

$$\leq (b-a) g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt + (b-a) f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt.$$

By the convexity of f and g, $\int_a^b g(x)dx > 0$, $f\left(\frac{a+b}{2}\right) \geq 0$ and (1.1), we get

$$(3.5)\quad (b-a) g\left(\frac{a+b}{2}\right) \geq \int_a^b g(t)dt > 0, \quad \int_a^b f(t)dt \geq (b-a) f\left(\frac{a+b}{2}\right) \geq 0.$$

Using (3.5), we obtain

$$(3.6)\quad (b-a)^2 f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) + \int_a^b f(t)dt \int_a^b g(t)dt$$
$$\geq (b-a) f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt + (b-a) f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt$$
$$= 2(b-a) f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt$$

and

$$(3.7)\quad (b-a) g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt + (b-a) f\left(\frac{a+b}{2}\right) \int_a^b g(t)dt$$
$$\leq 2(b-a) g\left(\frac{a+b}{2}\right) \int_a^b f(t)dt.$$

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Combining (3.4), (3.6) and (3.7), and dividing the combined formula by

$$2(b-a)g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$

yields (2.1).

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2.
(1) By Lemma 1.1 and the convexity of f and $-g$, we can see that $H(a,y;f)$ and $H(a,y;-g)$ both are nonnegative increasing with y on $[a,b]$. From the nonnegative increasing properties of f and g, we get that

$$\begin{aligned}F(a,y;f,g) &= g\left(\frac{a+y}{2}\right)\int_a^y f(t)dt - f\left(\frac{a+y}{2}\right)\int_a^y g(t)dt\\ &= g\left(\frac{a+y}{2}\right)\left(\int_a^y f(t)dt-(y-a)f\left(\frac{a+y}{2}\right)\right)\\ &\qquad + f\left(\frac{a+y}{2}\right)\left(\int_a^y g(t)dt-(y-a)g\left(\frac{a+y}{2}\right)\right)\\ &= g\left(\frac{a+y}{2}\right)\cdot H(a,y;f)+f\left(\frac{a+y}{2}\right)\cdot H(a,y;-g)\end{aligned}$$

is nonnegative increasing with y on $[a,b]$.

Since $F(a,y;f,g)$ is monotonically increasing with y on $[a,b]$, for any $y\in(a,b)$, we have

$$0 = F(a,a;f,g)\le F(a,y;f,g)\le F(a,b;f,g). \tag{3.8}$$

Expression (3.8) plus

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$

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yields

$$f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt \le f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt + F(a,y;f,g) \tag{3.9}$$
$$\le f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt + F(a,b;f,g)$$
$$= g\left(\frac{a+b}{2}\right)\int_a^b f(t)dt.$$

Expression (3.9) divided by

$$g\left(\frac{a+b}{2}\right)\int_a^b g(t)dt$$

yields (2.2).

(2) By Lemma 1.1 and the convexity of f and $-g$, we can see that $H(x,b;f)$ and $H(x,b;-g)$ are both nonnegative decreasing with x on $[a,b]$. Further, from the nonnegative decreasing properties of f and g, we obtain that

$$F(x,b;f,g) = g\left(\frac{x+b}{2}\right)\cdot H(x,b;f) + f\left(\frac{x+b}{2}\right)\cdot H(x,b;-g)$$

is nonnegative decreasing with x on $[a,b]$.

For any $x \in (a,b)$, then

$$0 = F(a,a;f,g) \le F(x,b;f,g) \le F(a,b;f,g). \tag{3.10}$$

Using (3.10), by the same arguments of proof for (1) of Theorem 2.2, we can also prove that (2.3) is true.

This completes the proof of Theorem 2.2. □

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