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p-VALENT MEROMORPHIC FUNCTIONS WITH ALTERNATING COEFFICIENTS BASED ON INTEGRAL OPERATOR

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ABSTRACT. By using a linear operator, a subclass of meromorphically $p-$valent functions with alternating coefficients is introduced. Some important properties of this class such as coefficient bounds, distortion bounds, etc. are found.

Key words and phrases: Meromorphic Functions, Alternating Coefficients, Distortion Bounds.

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1. INTRODUCTION

Let Σ_p be the class of functions of the form

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n, \quad A \geq 0 \tag{1.1}$$

that are regular in the punctured disk $\Delta^* = \{z : 0 < |z| < 1\}$ and σ_p be the subclass of Σ_p consisting of functions with alternating coefficients of the type

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} a_n z^n, \quad a_n \geq 0, \quad A \geq 0. \tag{1.2}$$

Let

$$\Sigma_p^*(\beta) = \left\{ f \in \Sigma_p : \mathrm{Re}\left(\frac{z[\mathcal{J}(f(z))]'}{\mathcal{J}(f(z))} \right) < -\beta, 0 \leq \beta < p \right\} \tag{1.3}$$

and let $\sigma_p^*(\beta) = \Sigma_p^*(\beta) \cap \sigma_p$ where

$$\mathcal{J}(f(z)) = (\gamma - p + 1) \int_0^1 (u^\gamma) f(uz) du, \quad p < \gamma \tag{1.4}$$

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is a linear operator.

With a simple calculation we obtain

$$\mathcal{J}(f(z)) = \begin{cases} Az^{-p} + \sum_{n=p}^{\infty} (-1)^{n-1} \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n, & f(z) \in \sigma_p; \\ Az^{-p} + \sum_{n=p}^{\infty} \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n, & f(z) \in \Sigma_p. \end{cases} \tag{1.5}$$

For more details about meromorphic p-valent functions, we can see the recent works of many authors in [1], [2], [3].

Also, Uralegaddi and Ganigi [4] worked on meromorphic univalent functions with alternating coefficients.

2. Coefficient Estimates

Theorem 2.1. *Let*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \in \Sigma_p.$$

If

$$\sum_{n=p}^{\infty} (n+\beta) \left(\frac{\gamma-p+1}{\gamma+n+1}\right) |a_n| \leq A(p-\beta), \tag{2.1}$$

then $f(z) \in \Sigma_p^*(\beta)$.

Proof. It is sufficient to show that

$$M = \left| \frac{\frac{z[\mathcal{J}f(z))]'}{\mathcal{J}f(z))} + p}{\frac{z[\mathcal{J}f(z))]'}{\mathcal{J}f(z))} - (p-2\beta)} \right| < 1 \quad \text{for} \quad |z| < 1.$$

However, by (1.5)

$$M = \left| \frac{-pAz^{-p} + \sum_{n=p}^{\infty} n \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n + pAz^{-p} + \sum_{n=p}^{\infty} p \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n}{-pAz^{-p} + \sum_{n=p}^{\infty} n \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n - (p-2\beta)Az^{-p} - \sum_{n=p}^{\infty} (p-2\beta) \left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n z^n} \right|$$

$$\leq \frac{\sum_{n=p}^{\infty} \left[(n+p)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\right] |a_n|}{2A(p-\beta) - \sum_{n=p}^{\infty} (n-p+2\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right) |a_n|}.$$

The last expression is less than or equal to 1 provided

$$\sum_{n=p}^{\infty} \left[(n+p)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\right] |a_n| \leq 2A(p-\beta) - \sum_{n=p}^{\infty} (n-p+2\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right) |a_n|,$$

which is equivalent to

$$\sum_{n=p}^{\infty} (n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right) |a_n| \leq A(p-\beta)$$

which is true by (2.1) so the proof is complete. □

The converse of Theorem 2.1 is also true for functions in $\sigma_p^*(\beta)$, where p is an odd number.

Theorem 2.2. *A function $f(z)$ in σ_p is in $\sigma_p^*(\beta)$ if and only if*

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \leq A(p-\beta). \tag{2.2}$$

Proof. According to Theorem 2.1 it is sufficient to prove the "only if" part. Suppose that

$$\operatorname{Re}\left(\frac{z(\mathcal{J}f(z))'}{(\mathcal{J}f(z))}\right)=\operatorname{Re}\left(\frac{-Apz^{-p}+\sum_{n=p}^{\infty}n(-1)^{n-1}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_nz^n}{Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_nz^n}\right)<-\beta. \tag{2.3}$$

By choosing values of z on the real axis so that $\frac{(z(\mathcal{J}f(z))'}{(\mathcal{J}f(z))}$ is real and clearing the denominator in (2.3) and letting $z \to -1$ through real values we obtain

$$Ap-\sum_{n=p}^{\infty}n\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \geq \beta\left(A+\sum_{n=p}^{\infty}\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n\right),$$

which is equivalent to

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_n \leq A(p-\beta).$$

□

Corollary 2.3. *If $f(z) \in \sigma_p^*(\beta)$ then*

$$a_n \leq \frac{A(p-\beta)(\gamma+n+1)}{(n+\beta)(\gamma-p+1)} \quad \textit{for} \quad n=p,p+1,\dots. \tag{2.4}$$

The result is sharp for functions of the type

$$f_n(z)=Az^{-p}+(-1)^{n-1}\frac{A(p-\beta)(\gamma+n+1)}{(n+\beta)(\gamma-p+1)}z^n. \tag{2.5}$$

3. DISTORTION BOUNDS AND IMPORTANT PROPERTIES OF $\sigma_p^*(\beta)$

In this section we obtain distortion bounds for functions in the class $\sigma_p^*(\beta)$ and prove some important properties of this class, where p is an odd number.

Theorem 3.1. *Let $f(z)=Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}a_nz^n$, $a_n \geq 0$ be in the class $\sigma_p^*(\beta)$ and $\beta \geq \gamma+1$ then*

$$Ar^{-p}-\frac{A(p-\beta)}{\gamma-p+1}r^p \leq |f(z)| \leq Ar^{-p}+\frac{A(p-\beta)}{\gamma-p+1}r^p. \tag{3.1}$$

Proof. Since $\beta \geq \gamma+1$, so $\frac{n+\beta}{\gamma+n+1} \geq 1$. Then

$$(\gamma-p+1)\sum_{n=p}^{\infty}a_n \leq \sum_{n=p}^{\infty}\left(\frac{n+\beta}{\gamma+n+1}\right)(\gamma-p+1)a_n \leq A(p-\beta),$$

and we have

$$|f(z)| = |Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1}a_n z^n| \\ \leq \frac{A}{r^p} + r^p \sum_{n=p}^{\infty} a_n \leq \frac{A}{r^p} + r^p \frac{A(p-\beta)}{(\gamma - p + 1)}.$$

Similarly,

$$|f(z)| \geq \frac{A}{r^p} - \sum_{n=p}^{\infty} a_n r^n \geq \frac{A}{r^p} - r^p \sum_{n=p}^{\infty} a_n \geq \frac{A}{r^p} - \frac{A(p-\beta)}{\gamma - p + 1} r^p.$$

□

Theorem 3.2. *Let*

$$f(z) = Az^{-p} + \sum_{n=p}^{\infty} a_n z^n \quad \textit{and} \quad g(z) = Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1} b_n z^n$$

be in the class $\sigma_p^*(\beta)$. Then the weighted mean of f and g defined by

$$W_j(z) = \frac{1}{2}[(1-j)f(z) + (1+j)g(z)]$$

is also in the same class.

Proof. Since f and g belong to $\sigma_p^*(\beta)$, then by (2.2) we have

$$\begin{cases} \sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right) a_n \leq A(p-\beta), \\ \sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right) b_n \leq A(p-\beta). \end{cases} \tag{3.2}$$

After a simple calculation we obtain

$$W_j(z) = Az^{-p} + \sum_{n=p}^{\infty}\left[\frac{1-j}{2}a_n + \frac{1+j}{2}b_n\right](-1)^{n-1} z^n.$$

However,

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right)\left[\frac{1-j}{2}a_n + \frac{1+j}{2}b_n\right] \\ = \left(\frac{1-j}{2}\right)\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right) a_n + \left(\frac{1+j}{2}\right)\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1)}{\gamma+n+1}\right) b_n \\ \overset{\text{by (3.2)}}{\leq} \left(\frac{1-j}{2}\right) A(p-\beta) + \left(\frac{1+j}{2}\right) A(p-\beta) \\ = A(p-\beta).$$

Hence by Theorem 2.2, $W_j(z) \in \sigma_p^*(\beta)$. □

Theorem 3.3. *Let*

$$f_k(z) = Az^{-p} + \sum_{n=p}^{\infty}(-1)^{n-1} a_{n,k} z^n \in \sigma_p^*(\beta), \qquad k = 1, 2, \ldots, m$$

then the arithmetic mean of $f_k(z)$ defined by

$$F(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z) \tag{3.3}$$

is also in the same class.

Proof. Since $f_k(z) \in \sigma_p^*(\beta)$, then by (2.2) we have

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)a_{n,k} \le A(p-\beta) \qquad (k=1,2,\dots,m). \tag{3.4}$$

After a simple calculation we obtain

$$\begin{aligned} F(z) &= \frac{1}{m}\sum_{k=1}^{m}\left(Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}a_{n,k}z^n\right) \\ &= Az^{-p}+\sum_{n=p}^{\infty}(-1)^{n-1}\left(\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right)z^n. \end{aligned}$$

However,

$$\sum_{n=p}^{\infty}(n+\beta)\left(\frac{\gamma-p+1}{\gamma+n+1}\right)\left(\frac{1}{m}\sum_{k=1}^{m}a_{n,k}\right) \overset{\text{by (3.4))}}{\le} \frac{1}{m}\sum_{k=1}^{m}A(p-\beta) = A(p-\beta)$$

which in view of Theorem 2.2 yields the proof of Theorem 3.3. □

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