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INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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ABSTRACT. In this paper we obtain new results concerning maximum modules of the polar derivative of a polynomial with restricted zeros. Our results generalize and refine upon the results of Aziz and Rather [3] and Jagjeet Kaur [9].

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. It was proved by Turán [11] that if $p(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

The result is best possible and equality holds in (1.1) if all the zeros of $p(z)$ lie on $|z| = 1$.

For the class of polynomials having all its zeros in $|z| \le k$, $k \ge 1$, Govil [7] proved:

Theorem A. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n *having all the zeros in* $|z| \le k$*,* $k \ge 1$*, then*

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^n} \max_{|z|=1} |p(z)| \,. \tag{1.2}$$

Inequality (1.2) *is sharp. Equality holds for* $p(z) = z^n + k^n$.

Let $D_\alpha\{p(z)\}$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to α, then

$$D_\alpha\{p(z)\} = np(z) + (\alpha - z)p'(z)\,.$$

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The polynomial $D_\alpha\{p(z)\}$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z).$$

Aziz and Rather [3] extended (1.2) to the polar derivative of a polynomial and proved the following:

Theorem B. *If the polynomial* $p(z)=\sum_{v=0}^{n}a_vz^v$ *has all its zeros in* $|z|\le k$, $k\ge 1$, *then for every real or complex number* α *with* $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha p(z)|\ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|. \tag{1.3}$$

The bound in Theorem B depends only on the zero of largest modulus and not on the other zeros even if some of them are close to the origin. Therefore, it would be interesting to obtain a bound, which depends on the location of all the zeros of a polynomial. In this connection we prove the following:

Theorem 1.1. *Let*

$$p(z)=\sum_{v=0}^{n}a_vz^v=a_n\prod_{v=1}^{n}(z-z_v),\quad a_n\ne 0,$$

be a polynomial of degree n, $|z_v|\le k_v$, $1\le v\le n$, *let* $k=\max(k_1,k_2,\dots,k_n)\ge 1$. *Then for every real or complex number* $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha p(z)| \ge (|\alpha|-k)\sum_{v=1}^{n}\frac{k}{k+k_v}\left[\frac{2}{1+k^n}\max_{|z|=1}|p(z)|+\frac{1}{k^n}\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|p(z)|\right]. \tag{1.4}$$

Dividing both sides of (1.4) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following refinement of a result due to Aziz [1].

Corollary 1.2. *Let* $p(z)=\sum_{v=0}^{n}a_vz^v=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\ne 0$, *be a polynomial of degree* n, $|z_v|\le k_v$, $1\le v\le n$, *let* $k=\max(k_1,k_2,\dots,k_n)\ge 1$. *Then,*

$$\max_{|z|=1}|p'(z)|\ge\sum_{v=1}^{n}\frac{k}{k+k_v}\left[\frac{2}{1+k^n}\max_{|z|=1}|p(z)|+\frac{1}{k^n}\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|p(z)|\right]. \tag{1.5}$$

Since $\frac{k}{k+k_v}\ge\frac{1}{2}$ for $1\le v\le n$, Theorem 1.1 gives the following result, which is an improvement of Theorem B.

Corollary 1.3. *If* $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\ne 0$, *is a polynomial of degree* n, *having all its zeros in* $|z|\le k$, $k\ge 1$, *then for every real or complex number* $|\alpha|\ge k$,

$$\max_{|z|=1}|D_\alpha p(z)|\ge n(|\alpha|-k)\left[\frac{1}{1+k^n}\max_{|z|=1}|p(z)|+\frac{1}{2k^n}\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|p(z)|\right]. \tag{1.6}$$

Dividing both sides of (1.6) by $|\alpha|$ and letting $|\alpha|\to\infty$, we obtain the following refinement of Theorem A due to Govil [7].

Corollary 1.4. *If* $p(z)=a_n\prod_{v=1}^{n}(z-z_v)$, $a_n\ne 0$, *is a polynomial of degree* n, *having all its zeros in* $|z|\le k$, $k\ge 1$, *then*

$$\max_{|z|=1}|p'(z)|\ge n\left[\frac{1}{1+k^n}\max_{|z|=1}|p(z)|+\frac{1}{2k^n}\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|p(z)|\right]. \tag{1.7}$$

The bound in Theorem 1.1 can be further improved for polynomials of degree $n \geq 2$. More precisely, we prove the following:

Theorem 1.5. *Let* $p(z) = \sum_{v=0}^{n} a_v z^v = a_n \prod_{v=1}^{n}(z - z_v), a_n \neq 0$*, be a polynomial of degree* $n \geq 2$*,* $|z_v| \leq k_v$*,* $1 \leq v \leq n$*, and let* $k = \max(k_1, k_2, \dots, k_n) \geq 1$*. Then for every real or complex number* $|\alpha| \geq k$*,*

$$\begin{aligned}(1.8)\quad \max_{|z|=1} |D_\alpha p(z)| \geq (|\alpha| - k) \sum_{v=1}^{n} \frac{k}{k + k_v} \Bigg[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left(\frac{k^n - 1}{k^n + 1} \right) \\ \times \min_{|z|=k} |p(z)| + \frac{2|a_{n-1}|}{k(1 + k^n)} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \Bigg] \\ + \left(1 - \frac{1}{k^2}\right) |na_0 + \alpha a_1| \quad \textit{for} \quad n > 2\end{aligned}$$

and

$$\begin{aligned}(1.9)\quad \max_{|z|=1} |D_\alpha p(z)| \geq (|\alpha| - k) \sum_{v=1}^{n} \frac{k}{k + k_v} \Bigg[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left(\frac{k^n - 1}{k^n + 1} \right) \\ \times \min_{|z|=k} |p(z)| + |a_1| \frac{(k - 1)^n}{k(1 + k^n)} \Bigg] + \left(1 - \frac{1}{k}\right) |na_0 + \alpha a_1| \quad \textit{for} \quad n = 2\,.\end{aligned}$$

Since $\frac{k}{k+k_v} \geq \frac{1}{2}$ for $1 \leq v \leq n$, the above theorem gives in particular:

Corollary 1.6. *If* $p(z) = a_n \prod_{v=1}^{n}(z - z_v)$*,* $a_n \neq 0$*, is a polynomial of degree* n *having all its zeros in* $|z| \leq k$*,* $k \geq 1$*, then for every real or complex number* $|\alpha| \geq k$*,*

$$\begin{aligned}(1.10)\quad \max_{|z|=1} |D_\alpha p(z)| \geq n(|\alpha| - k) \Bigg[\frac{1}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1}{2k^n} \left(\frac{k^n - 1}{k^n + 1} \right) \\ \times \min_{|z|=k} |p(z)| + \frac{|a_{n-1}|}{k(1 + k^n)} \left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2} \right) \Bigg] \\ + \left(1 - \frac{1}{k^2}\right) |na_0 + \alpha a_1| \quad \textit{for} \quad n > 2\,,\end{aligned}$$

and

$$\begin{aligned}(1.11)\quad \max_{|z|=1} |D_\alpha p(z)| \geq (|\alpha| - k) \Bigg[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left(\frac{k^n - 1}{k^n + 1} \right) \\ \times \min_{|z|=k} |p(z)| + \frac{|a_1|(k - 1)^n}{k(1 + k^n)} \Bigg] + \left(1 - \frac{1}{k}\right) |na_0 + \alpha a_1|, \quad \textit{for} \quad n = 2\,.\end{aligned}$$

Now it is easy to verify that if $k > 1$ and $n > 2$, then $\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2} \right) > 0$. Hence for polynomials of degree $n \geq 2$, the above corollary is a refinement of Theorem 1.1. In fact, except the case when $p(z)$ has all the zeros on $|z| = k$, $a_0 = 0$, $a_1 = 0$, and $a_{n-1} = 0$, the bound obtained by Theorem 1.5 is always sharper than the bound obtained by Theorem 1.1.

Remark 1. Dividing both sides of inequalities (1.8), (1.9), (1.10) and (1.11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we get the results due to Jagjeet Kaur [9]. In addition to this, if $\min_{|z|=k} |p(z)| = 0$ i.e. if a zero of a polynomial lies on $|z| = k$, then we obtain the results due to Govil [8].

Finally, as an application of Theorem 1.1 we prove the following:

Theorem 1.7. *If* $p(z) = \sum_{v=0}^{n} a_v z^v = a_n \prod_{v=1}^{n}(z - z_v)$, $a_n \neq 0$, *is a polynomial of degree* n, $|z_v| \geq k_v$, $1 \leq v \leq n$, *and* $k = \min(k_1, k_2, \dots, k_n) \leq 1$, *then for every real or complex number* δ *with* $|\delta| \leq k$,

$$\max_{|z|=1} |D_\delta p(z)| \geq (k - |\delta|)k^{n-1} \sum_{v=1}^{n} \frac{k_v}{k + k_v} \left[\frac{2}{1+k^n} \max_{|z|=1} |p(z)| + \frac{1-k^n}{k^n(1+k^n)} m \right], \tag{1.12}$$

where $m = \min_{|z|=k} |p(z)|$.

2. LEMMAS

For the proofs of the theorems, we need the following lemmas.

Lemma 2.1. *If* $p(z)$ *is a polynomial of degree* n, *then for* $R \geq 1$

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| . \tag{2.1}$$

The above lemma is a simple consequence of the Maximum Modulus Principle [10].

Lemma 2.2. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* n, *then for all* $R > 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - (R^n - R^{n-2})|p(0)| \quad \textit{for} \quad n \geq 2 \tag{2.2}$$

and

$$\max_{|z|=1} |p(z)| \leq R \max_{|z|=1} |p(z)| - (R-1)|p(0)| \quad \textit{for} \quad n = 1. \tag{2.3}$$

This result is due to Frappier, Rahman and Ruscheweyh [5].

Lemma 2.3. *If* $p(z) = a_n \prod_{v=1}^{n}(z - z_v)$, *is a polynomial of degree* $n \geq 2$, $|z_v| \geq 1$ *for* $1 \leq v \leq n$, *then*

$$\max_{|z|=R\geq 1} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)| - |a_1| \left(\frac{R^n-1}{n} - \frac{R^{n-2}}{n-2} \right) - \frac{R^n-1}{2} \min_{|z|=1} |p(z)|, \quad \textit{if} \quad n > 2 \tag{2.4}$$

and

$$\max_{|z|=R\geq 1} |p(z)| \leq \frac{R^2+1}{2} \max_{|z|=1} |p(z)| - |a_1| \frac{(R-1)^2}{2} - \frac{R^2-1}{2} \min_{|z|=1} |p(z)|, \quad \textit{if} \quad n = 2 . \tag{2.5}$$

The above lemma is due to Jagjeet Kaur [9].

Lemma 2.4. *If* $p(z) = a_n \prod_{v=1}^{n}(z - z_v)$, $a_n \neq 0$, *is a polynomial of degree* n, *such that* $|z_v| \leq 1$, $1 \leq v \leq n$, *then*

$$\max_{|z|=1} |p'(z)| \geq \sum_{v=1}^{n} \frac{1}{1+|z_v|} \max_{|z|=1} |p(z)|. \tag{2.6}$$

This lemma is due to Giroux, Rahman and Schmeisser [6].

Lemma 2.5. *If* $p(z)$ *is a polynomial of degree* n, *which has all its zeros in the disk* $|z| \leq k$, $k \geq 1$, *then*

$$\max_{|z|=k} |p(z)| \geq \frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)|. \tag{2.7}$$

Inequality (2.7) is best possible and equality holds for $p(z) = z^n + k^n$.

The above result is due to Aziz [1].

Lemma 2.6. *If $p(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \le k$, $k \ge 1$, then*

$$\max_{|z|=k}|p(z)| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)| + \frac{k^n-1}{1+k^n}\min_{|z|=k}|p(z)|. \tag{2.8}$$

The result is best possible and equality holds for $p(z) = z^n + k^n$.

Proof of Lemma 2.6. For $k = 1$, there is nothing to prove. Therefore it is sufficient to consider the case $k > 1$.

Let $m = \min_{|z|=k}|p(z)|$. Then $m \le |p(z)|$ for $|z| = k$.

Since all the zeros of $p(z)$ lie in $|z| \le k$, $k > 1$, by Rouche's theorem, for every λ with $|\lambda| < 1$, the polynomial $p(z) + \lambda m$ has all its zeros in $|z| \le k$, $k > 1$. Applying Lemma 2.5 to the polynomial $p(z) + \lambda m$, we get

$$\max_{|z|=k}|p(z) + \lambda m| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z) + \lambda m|.$$

Choosing the argument of λ such that $|p(z) + \lambda m| = |p(z)| + |\lambda| m$ and letting $|\lambda| \to 1$, we get

$$\max_{|z|=k}|p(z)| \ge \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)| + \frac{k^n-1}{1+k^n}\min_{|z|=k}|p(z)|.$$

This completes the proof of Lemma 2.6. □

Lemma 2.7. *If $p(z)$ is a polynomial of degree n and α is any real or complex number with $|\alpha| \ne 0$, then*

$$|D_\alpha q(z)| = |n\overline{\alpha}p(z) + (1-\overline{\alpha}z)p'(z)| \quad \textit{for } |z| = 1. \tag{2.9}$$

Lemma 2.7 is due to Aziz [2].

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. Let $G(z) = p(kz)$. Since the zeros of $p(z)$ are z_v, $1 \le v \le n$, the zeros of the polynomial $G(z)$ are z_v/k, $1 \le v \le n$, and because all the zeros of $p(z)$ lie in $|z| \le k$, all the zeros of $G(z)$ lie in $|z| \le 1$, therefore applying Lemma 2.4 to the polynomial $G(z)$, we get

$$\max_{|z|=1}|G'(z)| \ge \sum_{v=1}^{n}\frac{1}{1+\frac{|z_v|}{k}}\max_{|z|=1}|G(z)|. \tag{3.1}$$

Let $H(z) = z^n\overline{G(1/\overline{z})}$. Then it can be easily verified that

$$|H'(z)| = |nG(z) - zG'(z)|, \qquad \text{for } |z| = 1. \tag{3.2}$$

The polynomial $H(z)$ has all its zeros in $|z| \ge 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$, therefore, by the result of de Bruijn [4]

$$|H'(z)| \le |G'(z)| \text{ for } |z| = 1. \tag{3.3}$$

Now for every real or complex number α with $|\alpha| \ge k$, we have

$$\begin{aligned}|D_{\alpha/k}G(z)| &= \left|nG(z) - zG'(z) + \frac{\alpha}{k}G'(z)\right| \\ &\ge |\alpha/k||G'(z)| - |nG(z) - zG'(z)|.\end{aligned}$$

This gives with the help of (3.2) and (3.3) that

$$\max_{|z|=1}|D_{\alpha/k}G(z)| \ge \frac{|\alpha|-k}{k}\max_{|z|=1}|G'(z)|. \tag{3.4}$$

Using (3.1) in (3.4), we get

$$\max_{|z|=1} |D_{\alpha/k}G(z)| \geq \frac{|\alpha|-k}{k} \sum_{v=1}^{n} \frac{k}{k+|z_v|} \max_{|z|=1} |G(z)|.$$

Replacing $G(z)$ by $p(kz)$, we get

$$\max_{|z|=1} |D_{\alpha/k}p(kz)| \geq (|\alpha|-k) \sum_{v=1}^{n} \frac{1}{k+|z_v|} \max_{|z|=1} |p(kz)|$$

which implies

$$\max_{|z|=1} \left| np(kz) + \left(\frac{\alpha}{k} - z\right) kp'(kz) \right| \geq (|\alpha|-k) \sum_{v=1}^{n} \frac{1}{k+|z_v|} \max_{|z|=1} |p(kz)|,$$

which gives

$$\max_{|z|=k} |D_\alpha p(z)| \geq (|\alpha|-k) \sum_{v=1}^{n} \frac{1}{k+|z_v|} \max_{|z|=k} |p(z)|.$$

Using Lemma 2.6 in the above inequality, we get

$$\max_{|z|=k} |D_\alpha p(z)| \geq (|\alpha|-k) \sum_{v=1}^{n} \frac{1}{k+|z_v|} \left[\frac{2k^n}{1+k^n} \max_{|z|=1} |p(z)| + \left(\frac{k^n-1}{1+k^n}\right) \min_{|z|=k} |p(z)| \right]. \tag{3.5}$$

Since $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and $k \geq 1$, applying Lemma 2.1 to the polynomial $D_\alpha p(z)$, we get

$$\max_{|z|=k} |D_\alpha p(z)| \leq k^{n-1} \max_{|z|=1} |D_\alpha p(z)|. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{aligned} &\max_{|z|=1} |D_\alpha p(z)| \\ &\geq (|\alpha|-k) \sum_{v=1}^{n} \frac{k}{k+|z_v|} \left[\frac{2}{1+k^n} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left(\frac{k^n-1}{1+k^n}\right) \min_{|z|=k} |p(z)| \right] \\ &\geq (|\alpha|-k) \sum_{v=1}^{n} \frac{k}{k+k_v} \left[\frac{2}{1+k^n} \max_{|z|=1} |p(z)| + \frac{1}{k^n} \left(\frac{k^n-1}{1+k^n}\right) \min_{|z|=k} |p(z)| \right] \end{aligned}$$

which is the required result. Hence the proof Theorem 1.1 is complete. □

Proof of Theorem 1.5. Let $G(z) = p(kz)$. Since the zeros of $p(z)$ are z_v, $1 \leq v \leq n$, the zeros of the polynomial $G(z)$ are z_v/k, $1 \leq v \leq n$, and because all the zeros of $p(z)$ lie in $|z| \leq k$, all the zeros of $G(z)$ lie in $|z| \leq 1$, therefore applying Lemma 2.4 to the polynomial $G(z)$ and proceeding in the same way as in Theorem 1.1, we obtain

$$\max_{|z|=k} |D_\alpha p(z)| \geq (|\alpha|-k) \sum_{v=1}^{n} \frac{1}{k+|z_v|} \max_{|z|=k} |p(z)|. \tag{3.7}$$

Now let $q(z) = z^n p\left(\frac{1}{z}\right)$ be the reciprocal polynomial of $p(z)$. Since the polynomial $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$ the polynomial $q\left(\frac{z}{k}\right)$ has all its zeros in $|z| \geq 1$. Hence applying

(2.4) of Lemma 2.3 to the polynomial $q\left(\frac{z}{k}\right)$, $k \geq 1$, we get

$$\max_{|z|=k}\left|q\left(\frac{z}{k}\right)\right| \leq \frac{k^n+1}{2}\max_{|z|=1}\left|q\left(\frac{z}{k}\right)\right| - \left(\frac{k^n-1}{2}\right)\min_{|z|=1}\left|q\left(\frac{z}{k}\right)\right| - \frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right),$$

which gives

$$\max_{|z|=1}|p(z)| \leq \frac{k^n+1}{2k^n}\max_{|z|=k}|p(z)| - \left(\frac{k^n-1}{2k^n}\right)\min_{|z|=k}|p(z)| - \frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right),$$

which is equivalent to

$$\max_{|z|=k}|p(z)| \geq \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)| + \left(\frac{k^n-1}{1+k^n}\right)\min_{|z|=k}|p(z)| + \frac{2|a_{n-1}|k^{n-1}}{1+k^n}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right). \tag{3.8}$$

Using (3.8) in (3.7) we get

$$\max_{|z|=k}|D_\alpha p(z)| \geq (|\alpha|-k)\sum_{v=1}^{n}\frac{1}{k+|z_v|}\left[\frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)| + \left(\frac{k^n-1}{1+k^n}\right) \times \min_{|z|=k}|p(z)| + \frac{2|a_{n-1}|k^{n-1}}{1+k^n}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)\right] \quad \text{if} \quad n>2\,. \tag{3.9}$$

Since $D_\alpha p(z)$ is a polynomial of degree $n-1$ and $k \geq 1$, from (2.2) of Lemma 2.2, we get

$$\max_{|z|=k}|D_\alpha p(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha p(z)| - (k^{n-1}-k^{n-3})|D_\alpha p(0)|, \quad \text{if } n>2\,. \tag{3.10}$$

Combining (3.9) and (3.10) we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| &\geq (|\alpha|-k)\sum_{v=1}^{n}\frac{k}{k+|z_v|}\left[\frac{2}{1+k^n}\max_{|z|=1}|p(z)| + \frac{1}{k^n}\left(\frac{k^n-1}{1+k^n}\right)\min_{|z|=k}|p(z)|\right.\\ &\qquad \left. + \frac{2|a_{n-1}|}{k(1+k^n)}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)\right]\\ &\qquad + \left(1-\frac{1}{k^2}\right)|na_0+\alpha a_1|, \ \text{ if } \ n>2\\ &\geq (|\alpha|-k)\sum_{v=1}^{n}\frac{k}{k+k_v}\left[\frac{2}{1+k^n}\max_{|z|=1}|p(z)| + \frac{1}{k^n}\left(\frac{k^n-1}{1+k^n}\right)\min_{|z|=k}|p(z)|\right.\\ &\qquad \left. + \frac{2|a_{n-1}|}{k(1+k^n)}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)\right]\\ &\qquad + \left(1-\frac{1}{k^2}\right)|na_0+\alpha a_1|, \ \text{ if } \ n>2\end{aligned}$$

which completes the proof of (1.8).

The proof of (1.9) follows on the same lines as the proof of (1.8) but instead of (2.2) and (2.4) we use inequalities (2.3) and (2.5) respectively. We omit the details. □

Proof of Theorem 1.7. By hypothesis, the zeros of $p(z)$ satisfy $|z_v| \geq k_v$ for $1 \leq v \leq n$ such that $k = \min(k_1, k_2, \dots, k_n) \leq 1$. It follows that the zeros of the polynomial $q(z) = z^n \overline{p(1/\overline{z})}$ satisfy $1/|z_v| \leq 1/k_v$, $1 \leq v \leq n$ such that $1/k = \max(1/k_1, 1/k_2, \dots, 1/k_n) \geq 1$. On applying Theorem 1.1 to the polynomial $q(z)$, we get

$$\max_{|z|=1} |D_\alpha q(z)| \geq k^{n-1}(|\alpha| - 1/k) \sum_{v=1}^{n} \frac{1}{1/k + 1/k_v} \left[\frac{2/k^n}{1 + 1/k^n} \max_{|z|=1} |q(z)| + \frac{1/k^n - 1}{1 + 1/k^n} \min_{|z|=1/k} |q(z)| \right], \quad |\alpha| \geq \frac{1}{k}. \tag{3.11}$$

Now from Lemma 2.7 it follows that

$$|D_\alpha q(z)| = |\alpha||D_{1/\bar{\alpha}} p(z)| \text{ for } |z| = 1 .$$

Using the above equality in (3.11), we get for $|\alpha| \geq 1/k$,

$$|\alpha| \max_{|z|=1} |D_{1/\bar{\alpha}} p(z)| \geq k^{n-1}(|\alpha| - 1/k) \sum_{v=1}^{n} \frac{k k_v}{k + k_v} \left[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1 - k^n}{(1 + k^n)k^n} \min_{|z|=k} |p(z)| \right]. \tag{3.12}$$

Replacing $\frac{1}{\bar{\alpha}}$ by δ, so that $|\delta| \leq k$, we get from (3.12)

$$|1/\delta| \max_{|z|=1} |D_\delta p(z)| \geq k^{n-1}(|1/\delta| - 1/k) \sum_{v=1}^{n} \frac{k k_v}{k + k_v} \left[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1 - k^n}{(1 + k^n)k^n} \min_{|z|=k} |p(z)| \right],$$

or

$$\begin{aligned} \max_{|z|=1} |D_\delta p(z)| &\geq k^{n-1}(k - |\delta|) \sum_{v=1}^{n} \frac{k_v}{k + k_v} \left[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1 - k^n}{(1 + k^n)k^n} \min_{|z|=k} |p(z)| \right], \\ &\geq k^{n-1}(k - |\delta|) \sum_{v=1}^{n} \frac{k_v}{k + k_v} \left[\frac{2}{1 + k^n} \max_{|z|=1} |p(z)| + \frac{1 - k^n}{(1 + k^n)k^n} \min_{|z|=k} |p(z)| \right], \end{aligned}$$

which is (1.12). Hence the proof of Theorem 1.7 is complete. □

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