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ON SEVERAL NEW INEQUALITIES CLOSE TO HILBERT-PACHPATTE'S INEQUALITY

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ABSTRACT. In this paper we establish several new inequalities similar to Hilbert-Pachpatte's inequality. Moreover, some new generalizations of Hilbert-Pachpatte's inequality are presented.

Key words and phrases: Hilbert's inequality, Hölder's inequality, Jensen's inequality, Power mean inequality.

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1. INTRODUCTION

The well known Hardy-Hilbert's inequality is (see [1]):

Theorem 1.1. *If $p > 1$, $p' = p/(p-1)$ and $\sum_{m=1}^{\infty} a_m^p \le A$, $\sum_{n=1}^{\infty} b_n^{p'} \le B$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}(A)^{\frac{1}{p}}(B)^{\frac{1}{p'}}, \tag{1.1}$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

The integral analogue can be stated as follows:

Theorem 1.2. *If $p > 1$, $p' = p/(p-1)$ and $\int_0^{\infty} f^p(x)dx \le F$, $\int_0^{\infty} g^{p'}(y)dy \le G$, then*

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}F^{\frac{1}{p}}G^{\frac{1}{p'}} \tag{1.2}$$

unless $f \equiv 0$ or $g \equiv 0$.

The following two theorems were studied by Pachpatte (see [2])

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Theorem 1.3. *Let $p \geq 1$, $q \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0,x)$, $\tau \in (0,y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$, $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0,x)$, $t \in (0,y)$. Then*

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{s+t} dsdt \leq D(p,q,x,y) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds \right\}^{\frac{1}{2}} \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt \right\}^{\frac{1}{2}},$$

unless $f \equiv 0$ or $g \equiv 0$, where

$$D(p,q,x,y) = \frac{1}{2} pq\sqrt{xy}.$$

Theorem 1.4. *Let f, g, F, G be as in the above theorem, let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0,x)$, $\tau \in (0,y)$ and define $P(s) = \int_0^s p(\sigma)d\sigma$, $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in (0,x)$, $t \in (0,y)$, where x, y are positive real numbers. Let ϕ and ψ be real-valued, nonnegative, convex, and sub-multiplicative functions defined on $\mathbb{R}_+ = [0,\infty)$. Then*

$$\int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t} dsdt \leq L(x,y) \left\{ \int_0^x (x-s) \left[p(s)\phi\left(\frac{f(s)}{p(s)}\right) \right]^2 ds \right\}^{\frac{1}{2}} \times \left\{ \int_0^y (y-t) \left[q(t)\psi\left(\frac{g(t)}{q(t)}\right) \right]^2 dt \right\}^{\frac{1}{2}} \tag{1.3}$$

where

$$L(x,y) = \frac{1}{2} \left(\int_0^x \left[\frac{\phi(P(s))}{P(s)} \right]^2 ds \right)^{\frac{1}{2}} \left(\int_0^y \left[\frac{\psi(Q(t))}{Q(t)} \right]^2 dt \right)^{\frac{1}{2}}.$$

The inequalities in these theorems were studied extensively and numerous variants, generalizations, and extensions have appeared in the literature (see [3] – [9]).

The main purpose of the present article is to establish some new inequalities similar to the Hilbert-Pachpatte inequalities.

2. MAIN RESULTS

Lemma 2.1. *Suppose $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{w} = 1, \lambda > 0$, define the weight functions $\widetilde{\omega}_1(w,p,x), \widetilde{\omega}_2(w,p,x), \widetilde{\omega}_3(w,p,x)$ as*

$$\widetilde{\omega}_1(w,p,x) := \int_0^\infty \frac{1}{x^\lambda + y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \qquad x \in (0,\infty), \tag{2.1}$$

$$\widetilde{\omega}_2(w,p,x) := \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \qquad x \in (0,\infty), \tag{2.2}$$

$$\widetilde{\omega}_3(w,p,x) := \int_0^\infty \frac{\ln x/y}{x^\lambda - y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \qquad x \in (0,\infty). \tag{2.3}$$

Then

$$\widetilde{\omega}_1(w,p,x) = \frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)} x^{p(1-\frac{\lambda}{r})-1},$$

$$\widetilde{\omega}_2(w,p,x) = \frac{rw}{\lambda} x^{p(1-\frac{\lambda}{r})-1},$$

$$\widetilde{\omega}_3(w,p,x) = \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)}\right]^2 x^{p(1-\frac{\lambda}{r})-1}.$$

Proof. For fixed x, let $u = y^\lambda / x^\lambda$, then (2.1) turns into

$$\begin{aligned}\widetilde{\omega}_1(w,p,x) &= \frac{1}{\lambda} x^{p(1-\frac{\lambda}{r})-1} \int_0^\infty \frac{1}{1+u} u^{-1+\frac{1}{w}} du \\ &= \frac{1}{\lambda} x^{p(1-\frac{\lambda}{r})-1} B\left(\frac{1}{w}, \frac{1}{r}\right) \\ &= \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} x^{p(1-\frac{\lambda}{r})-1}\end{aligned}$$

and similarly, we can prove the others. □

Theorem 2.2. *Let* $m \geq 1$, $n \geq 1$, $p_i > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$ *for* $i = 0,1,2,3,4$, $p_0 = p$, $p_3 = k$, $p_4 = r$; $q_0 = q$, $q_3 = l$, $q_4 = w$ *and* $f(\sigma) \geq 0$, $g(\tau) \geq 0$. *Define* $F(s) = \int_0^s f(\sigma)d\sigma$ *and* $G(t) = \int_0^t g(\tau)d\tau$ *such that*

$$0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds < \infty, \qquad 0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt < \infty$$

for $\sigma, \tau, s, t \in (0,\infty)$. *Then*

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{F^m(s) G^n(t)}{(l s^{k/p_1} + k t^{l/p_2})(s^\lambda + t^\lambda)} ds dt \\ &\leq E_1(m,n,k,r,\lambda) \left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds\right\}^{\frac{1}{p}} \left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt\right\}^{\frac{1}{q}} \end{aligned} \tag{2.4}$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where* $E_1(m,n,k,r,\lambda) = \frac{\pi m n}{\lambda k l \sin(\pi/r)}$,

$$F_f(s) = \left\{\int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma\right\}^{\frac{1}{q_1}} \qquad \textit{and}$$

$$G_g(t) = \left\{\int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau\right\}^{\frac{1}{q_2}}.$$

Proof. From the hypotheses, we can easily observe that

$$F^m(s) = m \int_0^s F^{m-1}(\sigma) f(\sigma) d\sigma, \qquad s \in (0,\infty), \tag{2.5}$$

$$G^n(t) = n \int_0^t G^{n-1}(\tau) g(\tau) d\tau, \qquad t \in (0,\infty). \tag{2.6}$$

From (2.5) and (2.6), applying Hölder's inequality, equality (2.1) and Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \geq 0,\ b \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned}
&F^m(s)G^n(t)\\
&= mn\left(\int_0^s F^{m-1}(\sigma)f(\sigma)d\sigma\right)\left(\int_0^t G^{n-1}(\tau)g(\tau)d\tau\right)\\
&\leq mns^{1/p_1}t^{1/p_2}\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}d\sigma\right\}^{\frac{1}{q_1}}\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}d\tau\right\}^{\frac{1}{q_2}}\\
&\leq mn\left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l}\right)\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}d\sigma\right\}^{\frac{1}{q_1}}\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}d\tau\right\}^{\frac{1}{q_2}}.
\end{aligned}$$

Note that

$$F_f(s) = \left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}d\sigma\right\}^{\frac{1}{q_1}}, \qquad G_g(t) = \left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}d\tau\right\}^{\frac{1}{q_2}}.$$

Hence

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{k/p_1} + kt^{l/p_2})(s^\lambda + t^\lambda)}dsdt\\
&\leq \frac{mn}{kl}\int_0^\infty\int_0^\infty \frac{\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}d\sigma\right\}^{\frac{1}{q_1}}\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}d\tau\right\}^{\frac{1}{q_2}}}{s^\lambda + t^\lambda}dsdt\\
&= \frac{mn}{kl}\int_0^\infty\int_0^\infty \frac{1}{s^\lambda + t^\lambda}\left[F_f(s)\frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}}\right]\left[G_g(t)\frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}}\right]dsdt\\
&\leq \frac{mn}{kl}\left\{\int_0^\infty\int_0^\infty \frac{F_f^p(s)}{s^\lambda + t^\lambda}\cdot\frac{s^{(p-1)(1-\frac{\lambda}{r})}}{t^{(1-\frac{\lambda}{w})}}dsdt\right\}^{\frac{1}{p}}\left\{\int_0^\infty\int_0^\infty \frac{G_g^q(t)}{s^\lambda + t^\lambda}\cdot\frac{t^{(q-1)(1-\frac{\lambda}{w})}}{s^{(1-\frac{\lambda}{r})}}dsdt\right\}^{\frac{1}{q}}\\
&= \frac{mn}{kl}\left\{\int_0^\infty \widetilde{\omega}_1(w,p,s)F_f^p(s)ds\right\}^{\frac{1}{p}}\left\{\int_0^\infty \widetilde{\omega}_1(r,q,t)G_g^q(t)dt\right\}^{\frac{1}{q}}\\
&= \frac{\pi mn}{\lambda kl\sin(\pi/r)}\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)ds\right\}^{\frac{1}{p}}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)dt\right\}^{\frac{1}{q}}\\
&= E_1(m,n,k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)ds\right\}^{\frac{1}{p}}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)dt\right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let* $m \geq 1$, $n \geq 1$, $p_i > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$ *for* $i = 0,1,2,3,4$, $p_0 = p$, $p_3 = k$, $p_4 = r$; $q_0 = q$, $q_3 = l$, $q_4 = w$ *and* $f(\sigma) \geq 0$, $g(\tau) \geq 0$. *Define* $F(s) = \int_0^s f(\sigma)d\sigma$ *and* $G(t) = \int_0^t g(\tau)d\tau$ *such that* $0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)ds < \infty$, $0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)dt < \infty$ *for* $\sigma,\tau,s,t \in (0,\infty)$. *Then*

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{k/p_1} + kt^{l/p_2})\max\{s^\lambda, t^\lambda\}}dsdt\\
&\qquad\leq E_2(m,n,k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)ds\right\}^{\frac{1}{p}}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)dt\right\}^{\frac{1}{q}},
\end{aligned}$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where* $E_2(m,n,k,r,\lambda) = \frac{mnrw}{\lambda kl}$ *and*

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}},$$
$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$

Proof. By (2.5) and (2.6), using Hölder's inequality, equality (2.2) and Young's inequality: $ab \le \frac{a^p}{p} + \frac{b^q}{q}$, where $a \ge 0,\ b \ge 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1$. We obtain

$$\begin{aligned}
&F^m(s)G^n(t)\\
&= mn \left(\int_0^s F^{m-1}(\sigma) f(\sigma) d\sigma \right) \left(\int_0^t G^{n-1}(\tau) g(\tau) d\tau \right)\\
&\le mn s^{1/p_1} t^{1/p_2} \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}\\
&\le mn \left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l} \right) \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.
\end{aligned}$$

Note

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}},$$
$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$

Hence

$$\begin{aligned}
&\int_0^\infty \int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{k/p_1} + kt^{l/p_2}) \max\{s^\lambda, t^\lambda\}} ds dt\\
&\le \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{\left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}}{\max\{s^\lambda, t^\lambda\}} ds dt\\
&= \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{1}{\max\{s^\lambda, t^\lambda\}} \left[F_f(s) \frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}} \right] \left[G_g(t) \frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}} \right] ds dt\\
&\le \frac{mn}{kl} \left\{ \int_0^\infty \int_0^\infty \frac{F_f^p(s)}{\max\{s^\lambda, t^\lambda\}} \frac{s^{(p-1)(1-\frac{\lambda}{r})}}{t^{(1-\frac{\lambda}{w})}} ds dt \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \frac{G_g^q(t)}{\max\{s^\lambda, t^\lambda\}} \frac{t^{(q-1)(1-\frac{\lambda}{w})}}{s^{(1-\frac{\lambda}{r})}} ds dt \right\}^{\frac{1}{q}}\\
&= \frac{mn}{kl} \left\{ \int_0^\infty \widetilde{\omega}_2(w,p,s) F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \widetilde{\omega}_2(r,q,t) G_g^q(t) dt \right\}^{\frac{1}{q}}\\
&= \frac{mnrw}{\lambda kl} \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}}\\
&= E_2(m,n,k,r,\lambda) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Theorem 2.4. *Let* $m \geq 1$, $n \geq 1$, $p_i > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$ *for* $i = 0, 1, 2, 3, 4$, $p_0 = p$, $p_3 = k$, $p_4 = r$; $q_0 = q$, $q_3 = l$, $q_4 = w$ *and* $f(\sigma) \geq 0, g(\tau) \geq 0$. *Define* $F(s) = \int_0^s f(\sigma)d\sigma$ *and* $G(t) = \int_0^t g(\tau)d\tau$ *such that*

$$0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)ds < \infty,$$
$$0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)dt < \infty$$

for $\sigma, \tau, s, t \in (0, \infty)$. *Then*

$$\int_0^\infty \int_0^\infty \frac{\ln(s/t)}{(s^\lambda - t^\lambda)(ls^{k/p_1} + kt^{l/p_2})} F^m(s)G^n(t)dsdt \leq E_3(m, n, k, r, \lambda) \times \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)dt \right\}^{\frac{1}{q}},$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where* $E_3(m, n, k, r, \lambda) = \frac{\pi^2 mn}{\lambda^2 kl(\sin(\pi/r))^2}$,

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}},$$
$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$

Proof. From (2.5) and (2.6), by Hölder's inequality, equality (2.3) and Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $a \geq 0,\ b \geq 0,\ p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and using a similar method of proof to that of Theorem 2.2, the result can be clearly seen. □

Theorem 2.5. *Let* $m_i \geq 1$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, *and* $f_i(\sigma_i) \geq 0$ *for* $\sigma_i \in (0, x_i)$, *where* x_i *are positive real numbers and define* $F_i(t_i) = \int_0^{t_i} f_i(\sigma_i)d\sigma_i$, *for* $t_i \in (0, x_i)$, $i = 1, 2, \ldots, n$. *Then*

$$\int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \cdots dt_n \leq \prod_{i=1}^n D(m_i, x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\widetilde{F}_i(t_i) \right)^{q_i} dt_i \right\}^{\frac{1}{q_i}}, \tag{2.7}$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where* $D(m_i, x_i, p_i) = m_i x_i^{\frac{1}{p_i}}$, $\widetilde{F}_i(t_i) = F^{m_i-1}(t_i)f_i(t_i)$.

Proof. From the hypotheses, we know that

$$F_i^{m_i}(t_i) = m_i \int_0^{t_i} F_i^{(m_i-1)}(\sigma_i)f_i(\sigma_i)d\sigma_i, \qquad t_i \in (0, x_i). \tag{2.8}$$

Then

$$\prod_{i=1}^n F_i^{m_i}(t_i) = \prod_{i=1}^n m_i \int_0^{t_i} F_i^{m_i-1}(\sigma_i)f_i(\sigma_i)d\sigma_i. \tag{2.9}$$

Using Hölder's inequality, we have

$$\int_0^{t_i} F_i^{m_i-1}(\sigma_i)d\sigma_i \le t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} (F_i^{m_i-1}(\sigma_i) f_i(\sigma_i))^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}} \triangleq t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} \left(\widetilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}. \tag{2.10}$$

By (2.9) and (2.10) it follows that

$$\prod_{i=1}^n F_i^{m_i}(t_i) \le \prod_{i=1}^n m_i t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} \left(\widetilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$

Applying Young's inequality

$$\prod_{k=1}^n |a_k| \le \sum_{k=1}^n \frac{1}{p_k} |a_k|^{p_k},$$

where $1 < p_k < \infty$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, we observe that

$$\frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} \le \prod_{i=1}^n m_i \left\{ \int_0^{t_i} \left(\widetilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$

Integrating over t_i from 0 to x_i where i runs from 1 to n, and using the Hölder's inequality, we get

$$\begin{aligned} \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \cdots dt_n & \\ \le \prod_{i=1}^n m_i \left[\int_0^{x_i} \left(\int_0^{t_i} \left(\widetilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right)^{\frac{1}{q_i}} dt_i \right] & \\ \le \prod_{i=1}^n m_i x_i^{\frac{1}{p_i}} \left[\int_0^{x_i} \left(\int_0^{t_i} \left(\widetilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right) dt_i \right]^{\frac{1}{q_i}} & \\ = \prod_{i=1}^n D(m_i, x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\widetilde{F}_i(t_i) \right)^{q_i} dt_i \right\}^{\frac{1}{q_i}} & \end{aligned}$$

This completes the proof. □

Remark 2.6. In the special case when $p_1 = p_2 = \cdots = p_n = n$, the inequality (2.7) reduces to the following inequality,

$$\begin{aligned} \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n t_i} dt_1 dt_2 \cdots dt_n & \\ \le \frac{1}{n} \prod_{i=1}^n \overline{D}(m_i, x_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\widetilde{F}_i(t_i) \right)^{\frac{n}{n-1}} dt_i \right\}^{\frac{n-1}{n}} & \end{aligned} \tag{2.11}$$

where $\overline{D}(m_i, x_i) = m_i x_i^{\frac{1}{n}}$. Moreover, (i) (2.11) reduces to Theorem 1.3 which belongs to Pachpatte for $n = 2$; (ii) when $m_1 = m_2 = \cdots = m_n = 1$, (2.11) turns into

$$\begin{aligned} \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n F_i(t_i)}{\sum_{i=1}^n t_i} dt_1 dt_2 \cdots dt_n \\ \leq \frac{1}{n}\prod_{i=1}^n \overline{D}(m_i, x_i)\left\{\int_0^{x_i}(x_i - t_i)(f_i(t_i))^{\frac{n}{n-1}} dt_i\right\}^{\frac{n-1}{n}} \end{aligned} \tag{2.12}$$

Theorem 2.7. *Let f_i, F_i be as in the above theorem, $p > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, let $p_i(\sigma_i)$ be positive function defined for $\sigma_i \in (0, x_i)$, and define $P_i(t_i) = \int_0^{t_i} p_i(\sigma_i)d\sigma_i$ for $t_i \in (0, x_i)$,where x_i are positive real numbers. Let ϕ_i be real-valued, nonnegative, convex, and sub-multiplicative function defined on $R_+ = [0, \infty)$, $i = 1, 2, \ldots, n$. Then*

$$\begin{aligned} \int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \cdots dt_n \\ \leq \prod_{i=1}^n L(x_i, p_i)\left\{\int_0^{x_i}(x_i - t_i)\left[p_i(t_i)\phi_i\left(\frac{f_i(t_i)}{p_i(t_i)}\right)\right]^{q_i} dt_i\right\}^{\frac{1}{q_i}}, \end{aligned} \tag{2.13}$$

where

$$L(x_i, p_i) = \left(\int_0^{x_i}\left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)}\right]^{p_i} dt_i\right)^{\frac{1}{p_i}}.$$

Proof. Applying Jensen's inequality and Hölder's inequality, it is clear to observe that

$$\begin{aligned} \phi_i(F_i(t_i)) &= \phi_i\left(\frac{P_i(t_i)\int_0^{t_i} p_i(\sigma_i)\frac{f_i(\sigma_i)}{p_i(\sigma_i)}d\sigma_i}{\int_0^{t_i} p_i(\sigma_i)d\sigma_i}\right) \\ &\leq \frac{\phi_i(P_i(t_i))}{P_i(t_i)}\int_0^{t_i} p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)d\sigma_i \\ &\leq \frac{\phi_i(P_i(t_i))}{P_i(t_i)} t_i^{\frac{1}{p_i}}\left\{\int_0^{t_i}\left[p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right]^{q_i} d\sigma_i\right\}^{\frac{1}{q_i}}. \end{aligned}$$

Using Young's inequality, we obtain that

$$\frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}} \leq \prod_{i=1}^n \frac{\phi_i(P_i(t_i))}{P_i(t_i)}\left\{\int_0^{t_i}\left[p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right]^{q_i} d\sigma_i\right\}^{\frac{1}{q_i}}.$$

Integrating both sides of the above inequality over t_i from 0 to x_i with i running from i to n, and applying the Hölder inequality, we get

$$\int_0^{x_1}\int_0^{x_2}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}}dt_1dt_2\cdots dt_n$$

$$\leq \prod_{i=1}^n\left[\int_0^{x_i}\frac{\phi_i(P_i(t_i))}{P_i(t_i)}\left(\int_0^{t_i}\left[p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right]^{q_i}d\sigma_i\right)^{\frac{1}{q_i}}dt_i\right]$$

$$\leq \prod_{i=1}^n\left\{\int_0^{x_i}\left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)}\right]^{p_i}dt_i\right\}^{\frac{1}{p_i}}\left\{\int_0^{x_i}\left(\int_0^{t_i}\left[p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right]^{q_i}d\sigma_i\right)dt_i\right\}^{\frac{1}{q_i}}$$

$$= \prod_{i=1}^n\left\{\int_0^{x_i}\left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)}\right]^{p_i}dt_i\right\}^{\frac{1}{p_i}}\left\{\int_0^{x_i}(x_i-t_i)\left[p_i(t_i)\phi_i\left(\frac{f_i(t_i)}{p_i(t_i)}\right)\right]^{q_i}dt_i\right\}^{\frac{1}{q_i}}$$

$$= \prod_{i=1}^n L(x_i,p_i)\left\{\int_0^{x_i}(x_i-t_i)\left[p_i(t_i)\phi_i\left(\frac{f_i(t_i)}{p_i(t_i)}\right)\right]^{q_i}dt_i\right\}^{\frac{1}{q_i}}$$

This completes the theorem. □

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