

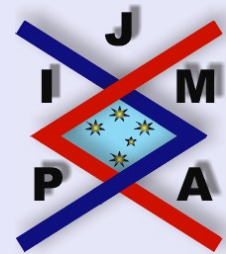
ON SEVERAL NEW INEQUALITIES CLOSE TO HILBERT-PACHPATTE'S INEQUALITY

BING HE AND YONGJIN LI

Department of Mathematics
Sun Yat-sen University
Guangzhou, 510275 P. R. China

EMail: hzs314@163.com

EMail: stslyj@mail.sysu.edu.cn



volume 7, issue 4, article 154,
2006.

*Received 27 April, 2006;
accepted 09 October, 2006.*

Communicated by: B. Yang

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Abstract

In this paper we establish several new inequalities similar to Hilbert-Pachpatte's inequality. Moreover, some new generalizations of Hilbert-Pachpatte's inequality are presented.

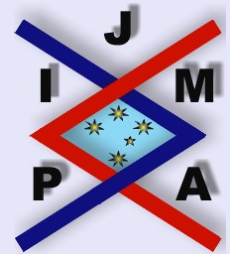
2000 Mathematics Subject Classification: 26D15.

Key words: Hilbert's inequality, Hölder's inequality, Jensen's inequality, Power mean inequality.

The work was partially supported by the Foundation of Sun Yat-sen University Advanced Research Centre.

Contents

1	Introduction	3
2	Main Results	6
	References	



**On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality**

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 2 of 20

1. Introduction

The well known Hardy-Hilbert's inequality is (see [1]):

Theorem 1.1. *If $p > 1$, $p' = p/(p - 1)$ and $\sum_{m=1}^{\infty} a_m^p \leq A$, $\sum_{n=1}^{\infty} b_n^{p'} \leq B$, then*

$$(1.1) \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} (A)^{\frac{1}{p}} (B)^{\frac{1}{p'}},$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

The integral analogue can be stated as follows:

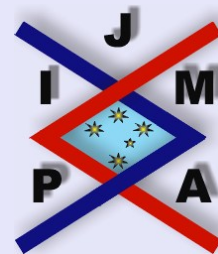
Theorem 1.2. *If $p > 1$, $p' = p/(p - 1)$ and $\int_0^{\infty} f^p(x)dx \leq F$, $\int_0^{\infty} g^{p'}(y)dy \leq G$, then*

$$(1.2) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} F^{\frac{1}{p}} G^{\frac{1}{p'}}$$

unless $f \equiv 0$ or $g \equiv 0$.

The following two theorems were studied by Pachpatte (see [2])

Theorem 1.3. *Let $p \geq 1$, $q \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$,*



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 3 of 20

$G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0, x)$, $t \in (0, y)$. Then

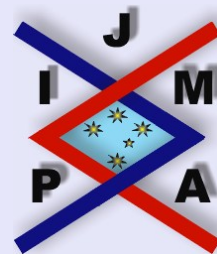
$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{s+t} dsdt \leq D(p, q, x, y) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds \right\}^{\frac{1}{2}} \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt \right\}^{\frac{1}{2}},$$

unless $f \equiv 0$ or $g \equiv 0$, where

$$D(p, q, x, y) = \frac{1}{2}pq\sqrt{xy}.$$

Theorem 1.4. Let f, g, F, G be as in the above theorem, let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0, x)$, $\tau \in (0, y)$ and define $P(s) = \int_0^s p(\sigma)d\sigma$, $Q(t) = \int_0^t q(\tau)d\tau$ for $s \in (0, x)$, $t \in (0, y)$, where x, y are positive real numbers. Let ϕ and ψ be real-valued, nonnegative, convex, and sub-multiplicative functions defined on $\mathbb{R}_+ = [0, \infty)$. Then

$$(1.3) \int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t} dsdt \leq L(x, y) \left\{ \int_0^x (x-s) \left[p(s)\phi\left(\frac{f(s)}{p(s)}\right) \right]^2 ds \right\}^{\frac{1}{2}} \times \left\{ \int_0^y (y-t) \left[q(t)\psi\left(\frac{g(t)}{q(t)}\right) \right]^2 dt \right\}^{\frac{1}{2}}$$



On Several New Inequalities Close to Hilbert-Pachpatte's Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

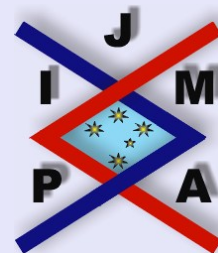
Page 4 of 20

where

$$L(x, y) = \frac{1}{2} \left(\int_0^x \left[\frac{\phi(P(s))}{P(s)} \right]^2 ds \right)^{\frac{1}{2}} \left(\int_0^y \left[\frac{\psi(Q(t))}{Q(t)} \right]^2 dt \right)^{\frac{1}{2}} .$$

The inequalities in these theorems were studied extensively and numerous variants, generalizations, and extensions have appeared in the literature (see [3] – [9]).

The main purpose of the present article is to establish some new inequalities similar to the Hilbert-Pachpatte inequalities.



**On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality**

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 5 of 20

2. Main Results

Lemma 2.1. Suppose $p > 1, \frac{1}{p} + \frac{1}{q} = 1, r > 1, \frac{1}{r} + \frac{1}{w} = 1, \lambda > 0$, define the weight functions $\tilde{\omega}_1(w, p, x), \tilde{\omega}_2(w, p, x), \tilde{\omega}_3(w, p, x)$ as

$$(2.1) \quad \tilde{\omega}_1(w, p, x) := \int_0^\infty \frac{1}{x^\lambda + y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \quad x \in (0, \infty),$$

$$(2.2) \quad \tilde{\omega}_2(w, p, x) := \int_0^\infty \frac{1}{\max\{x^\lambda, y^\lambda\}} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \quad x \in (0, \infty),$$

$$(2.3) \quad \tilde{\omega}_3(w, p, x) := \int_0^\infty \frac{\ln x/y}{x^\lambda - y^\lambda} \cdot \frac{x^{(p-1)(1-\frac{\lambda}{r})}}{y^{1-\frac{\lambda}{w}}} dy, \quad x \in (0, \infty).$$

Then

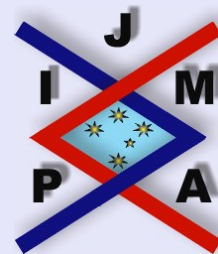
$$\tilde{\omega}_1(w, p, x) = \frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)} x^{p(1-\frac{\lambda}{r})-1},$$

$$\tilde{\omega}_2(w, p, x) = \frac{rw}{\lambda} x^{p(1-\frac{\lambda}{r})-1},$$

$$\tilde{\omega}_3(w, p, x) = \left[\frac{\pi}{\lambda \sin\left(\frac{\pi}{r}\right)} \right]^2 x^{p(1-\frac{\lambda}{r})-1}.$$

Proof. For fixed x , let $u = y^\lambda/x^\lambda$, then (2.1) turns into

$$\tilde{\omega}_1(w, p, x) = \frac{1}{\lambda} x^{p(1-\frac{\lambda}{r})-1} \int_0^\infty \frac{1}{1+u} u^{-1+\frac{1}{w}} du$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 6 of 20

$$= \frac{1}{\lambda} x^{p(1-\frac{\lambda}{r})-1} B\left(\frac{1}{w}, \frac{1}{r}\right) = \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} x^{p(1-\frac{\lambda}{r})-1}$$

and similarly, we can prove the others. \square

Theorem 2.2. Let $m \geq 1, n \geq 1, p_i > 1, \frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 0, 1, 2, 3, 4$, $p_0 = p, p_3 = k, p_4 = r; q_0 = q, q_3 = l, q_4 = w$ and $f(\sigma) \geq 0, g(\tau) \geq 0$. Define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$ such that

$$0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds < \infty, \quad 0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt < \infty$$

for $\sigma, \tau, s, t \in (0, \infty)$. Then

$$(2.4) \quad \int_0^\infty \int_0^\infty \frac{F^m(s) G^n(t)}{(ls^{k/p_1} + kt^{l/p_2})(s^\lambda + t^\lambda)} ds dt$$

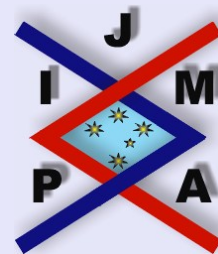
$$\leq E_1(m, n, k, r, \lambda) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}}$$

unless $f \equiv 0$ or $g \equiv 0$, where $E_1(m, n, k, r, \lambda) = \frac{\pi mn}{\lambda kl \sin(\pi/r)}$,

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \quad \text{and}$$

$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 7 of 20

Proof. From the hypotheses, we can easily observe that

$$(2.5) \quad F^m(s) = m \int_0^s F^{m-1}(\sigma) f(\sigma) d\sigma, \quad s \in (0, \infty),$$

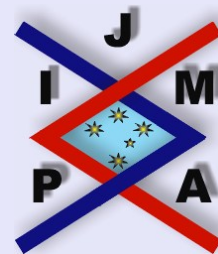
$$(2.6) \quad G^n(t) = n \int_0^t G^{n-1}(\tau) g(\tau) d\tau, \quad t \in (0, \infty).$$

From (2.5) and (2.6), applying Hölder's inequality, equality (2.1) and Young's inequality:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \geq 0$, $b \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, we obtain

$$\begin{aligned} & F^m(s)G^n(t) \\ &= mn \left(\int_0^s F^{m-1}(\sigma) f(\sigma) d\sigma \right) \left(\int_0^t G^{n-1}(\tau) g(\tau) d\tau \right) \\ &\leq mn s^{1/p_1} t^{1/p_2} \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}} \\ &\leq mn \left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l} \right) \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \\ &\quad \times \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}. \end{aligned}$$



**On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality**

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

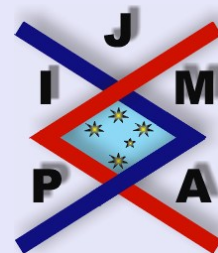
Page 8 of 20

Note that

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}}, \quad G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$

Hence

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{k/p_1} + kt^{l/p_2})(s^\lambda + t^\lambda)} ds dt \\ & \leq \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{\left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}}{s^\lambda + t^\lambda} ds dt \\ & = \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{1}{s^\lambda + t^\lambda} \left[F_f(s) \frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}} \right] \left[G_g(t) \frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}} \right] ds dt \\ & \leq \frac{mn}{kl} \left\{ \int_0^\infty \int_0^\infty \frac{F_f^p(s)}{s^\lambda + t^\lambda} \cdot \frac{s^{(p-1)(1-\frac{\lambda}{r})}}{t^{(1-\frac{\lambda}{w})}} ds dt \right\}^{\frac{1}{p}} \\ & \quad \times \left\{ \int_0^\infty \int_0^\infty \frac{G_g^q(t)}{s^\lambda + t^\lambda} \cdot \frac{t^{(q-1)(1-\frac{\lambda}{w})}}{s^{(1-\frac{\lambda}{r})}} ds dt \right\}^{\frac{1}{q}} \\ & = \frac{mn}{kl} \left\{ \int_0^\infty \tilde{\omega}_1(w, p, s) F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \tilde{\omega}_1(r, q, t) G_g^q(t) dt \right\}^{\frac{1}{q}} \\ & = \frac{\pi mn}{\lambda kl \sin(\pi/r)} \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}} \end{aligned}$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 9 of 20

$$= E_1(m, n, k, r, \lambda) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}}.$$

This completes the proof. \square

Theorem 2.3. Let $m \geq 1, n \geq 1, p_i > 1, \frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 0, 1, 2, 3, 4, p_0 = p, p_3 = k, p_4 = r; q_0 = q, q_3 = l, q_4 = w$ and $f(\sigma) \geq 0, g(\tau) \geq 0$. Define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$ such that $0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds < \infty, 0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt < \infty$ for $\sigma, \tau, s, t \in (0, \infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{F^m(s) G^n(t)}{(ls^{k/p_1} + kt^{l/p_2}) \max\{s^\lambda, t^\lambda\}} ds dt$$

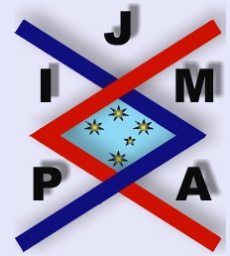
$$\leq E_2(m, n, k, r, \lambda) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}},$$

unless $f \equiv 0$ or $g \equiv 0$, where $E_2(m, n, k, r, \lambda) = \frac{mnrw}{\lambda kl}$ and

$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}},$$

$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 10 of 20

Proof. By (2.5) and (2.6), using Hölder's inequality, equality (2.2) and Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $a \geq 0$, $b \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. We obtain

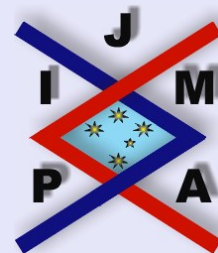
$$\begin{aligned} & F^m(s)G^n(t) \\ &= mn \left(\int_0^s F^{m-1}(\sigma)f(\sigma)d\sigma \right) \left(\int_0^t G^{n-1}(\tau)g(\tau)d\tau \right) \\ &\leq mns^{1/p_1}t^{1/p_2} \left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}} \\ &\leq mn \left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l} \right) \left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \\ &\quad \times \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}. \end{aligned}$$

Note

$$\begin{aligned} F_f(s) &= \left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}}, \\ G_g(t) &= \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}. \end{aligned}$$

Hence

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{F^m(s)G^n(t)}{(ls^{k/p_1} + kt^{l/p_2}) \max\{s^\lambda, t^\lambda\}} dsdt \\ &\leq \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{\left\{ \int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}} \left\{ \int_0^t (G^{n-1}(\tau)g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}}{\max\{s^\lambda, t^\lambda\}} dsdt \end{aligned}$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 11 of 20

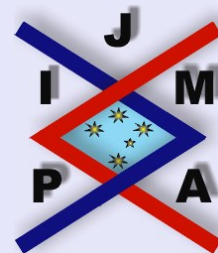
$$\begin{aligned}
&= \frac{mn}{kl} \int_0^\infty \int_0^\infty \frac{1}{\max\{s^\lambda, t^\lambda\}} \left[F_f(s) \frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}} \right] \left[G_g(t) \frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}} \right] dsdt \\
&\leq \frac{mn}{kl} \left\{ \int_0^\infty \int_0^\infty \frac{F_f^p(s)}{\max\{s^\lambda, t^\lambda\}} \frac{s^{(p-1)(1-\frac{\lambda}{r})}}{t^{(1-\frac{\lambda}{w})}} dsdt \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_0^\infty \int_0^\infty \frac{G_g^q(t)}{\max\{s^\lambda, t^\lambda\}} \frac{t^{(q-1)(1-\frac{\lambda}{w})}}{s^{(1-\frac{\lambda}{r})}} dsdt \right\}^{\frac{1}{q}} \\
&= \frac{mn}{kl} \left\{ \int_0^\infty \tilde{\omega}_2(w, p, s) F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \tilde{\omega}_2(r, q, t) G_g^q(t) dt \right\}^{\frac{1}{q}} \\
&= \frac{mnrw}{\lambda kl} \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}} \\
&= E_2(m, n, k, r, \lambda) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. \square

Theorem 2.4. Let $m \geq 1, n \geq 1, p_i > 1, \frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 0, 1, 2, 3, 4$, $p_0 = p, p_3 = k, p_4 = r; q_0 = q, q_3 = l, q_4 = w$ and $f(\sigma) \geq 0, g(\tau) \geq 0$. Define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$ such that

$$0 < \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds < \infty,$$

$$0 < \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt < \infty$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 12 of 20

for $\sigma, \tau, s, t \in (0, \infty)$. Then

$$\int_0^\infty \int_0^\infty \frac{\ln(s/t)}{(s^\lambda - t^\lambda)(ls^{k/p_1} + kt^{l/p_2})} F^m(s) G^n(t) ds dt \leq E_3(m, n, k, r, \lambda) \\ \times \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s) ds \right\}^{\frac{1}{p}} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t) dt \right\}^{\frac{1}{q}},$$

unless $f \equiv 0$ or $g \equiv 0$, where $E_3(m, n, k, r, \lambda) = \frac{\pi^2 mn}{\lambda^2 kl (\sin(\pi/r))^2}$,

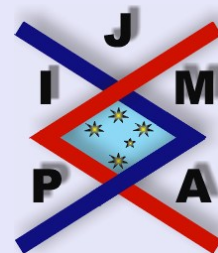
$$F_f(s) = \left\{ \int_0^s (F^{m-1}(\sigma) f(\sigma))^{q_1} d\sigma \right\}^{\frac{1}{q_1}},$$

$$G_g(t) = \left\{ \int_0^t (G^{n-1}(\tau) g(\tau))^{q_2} d\tau \right\}^{\frac{1}{q_2}}.$$

Proof. From (2.5) and (2.6), by Hölder's inequality, equality (2.3) and Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $a \geq 0$, $b \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and using a similar method of proof to that of Theorem 2.2, the result can be clearly seen. \square

Theorem 2.5. Let $m_i \geq 1$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, and $f_i(\sigma_i) \geq 0$ for $\sigma_i \in (0, x_i)$, where x_i are positive real numbers and define $F_i(t_i) = \int_0^{t_i} f_i(\sigma_i) d\sigma_i$, for $t_i \in (0, x_i)$, $i = 1, 2, \dots, n$. Then

$$(2.7) \quad \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \dots dt_n$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 13 of 20

$$\leq \prod_{i=1}^n D(m_i, x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\tilde{F}_i(t_i) \right)^{q_i} dt_i \right\}^{\frac{1}{q_i}},$$

unless $f \equiv 0$ or $g \equiv 0$, where $D(m_i, x_i, p_i) = m_i x_i^{\frac{1}{p_i}}$, $\tilde{F}_i(t_i) = F_i^{m_i-1}(t_i) f_i(t_i)$.

Proof. From the hypotheses, we know that

$$(2.8) \quad F_i^{m_i}(t_i) = m_i \int_0^{t_i} F_i^{(m_i-1)}(\sigma_i) f_i(\sigma_i) d\sigma_i, \quad t_i \in (0, x_i).$$

Then

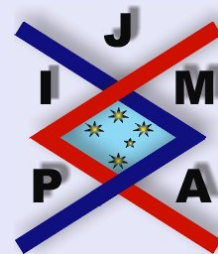
$$(2.9) \quad \prod_{i=1}^n F_i^{m_i}(t_i) = \prod_{i=1}^n m_i \int_0^{t_i} F_i^{m_i-1}(\sigma_i) f_i(\sigma_i) d\sigma_i.$$

Using Hölder's inequality, we have

$$(2.10) \quad \int_0^{t_i} F_i^{m_i-1}(\sigma_i) d\sigma_i \leq t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} (F_i^{m_i-1}(\sigma_i) f_i(\sigma_i))^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}} \\ \triangleq t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} \left(\tilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$

By (2.9) and (2.10) it follows that

$$\prod_{i=1}^n F_i^{m_i}(t_i) \leq \prod_{i=1}^n m_i t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} \left(\tilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 14 of 20

Applying Young's inequality

$$\prod_{k=1}^n |a_k| \leq \sum_{k=1}^n \frac{1}{p_k} |a_k|^{p_k},$$

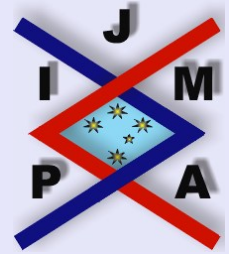
where $1 < p_k < \infty$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, we observe that

$$\frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} \leq \prod_{i=1}^n m_i \left\{ \int_0^{t_i} \left(\tilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$

Integrating over t_i from 0 to x_i where i runs from 1 to n , and using the Hölder's inequality, we get

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \cdots dt_n \\ & \leq \prod_{i=1}^n m_i \left[\int_0^{x_i} \left(\int_0^{t_i} \left(\tilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right)^{\frac{1}{q_i}} dt_i \right] \\ & \leq \prod_{i=1}^n m_i x_i^{\frac{1}{p_i}} \left[\int_0^{x_i} \left(\int_0^{t_i} \left(\tilde{F}_i(\sigma_i) \right)^{q_i} d\sigma_i \right) dt_i \right]^{\frac{1}{q_i}} \\ & = \prod_{i=1}^n D(m_i, x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\tilde{F}_i(t_i) \right)^{q_i} dt_i \right\}^{\frac{1}{q_i}} \end{aligned}$$

This completes the proof. \square



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 15 of 20

Remark 1. In the special case when $p_1 = p_2 = \dots = p_n = n$, the inequality (2.7) reduces to the following inequality,

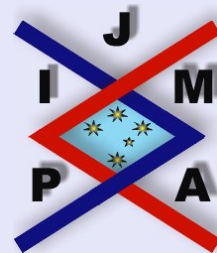
$$(2.11) \quad \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{m_i}(t_i)}{\sum_{i=1}^n t_i} dt_1 dt_2 \dots dt_n \\ \leq \frac{1}{n} \prod_{i=1}^n \overline{D}(m_i, x_i) \left\{ \int_0^{x_i} (x_i - t_i) \left(\widetilde{F}_i(t_i) \right)^{\frac{n}{n-1}} dt_i \right\}^{\frac{n-1}{n}}$$

where $\overline{D}(m_i, x_i) = m_i x_i^{\frac{1}{n}}$. Moreover, (i) (2.11) reduces to Theorem 1.3 which belongs to Pachpatte for $n = 2$; (ii) when $m_1 = m_2 = \dots = m_n = 1$, (2.11) turns into

$$(2.12) \quad \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \frac{\prod_{i=1}^n F_i(t_i)}{\sum_{i=1}^n t_i} dt_1 dt_2 \dots dt_n \\ \leq \frac{1}{n} \prod_{i=1}^n \overline{D}(m_i, x_i) \left\{ \int_0^{x_i} (x_i - t_i) (f_i(t_i))^{\frac{n}{n-1}} dt_i \right\}^{\frac{n-1}{n}}$$

Theorem 2.6. Let f_i, F_i be as in the above theorem, $p > 1$, $\frac{1}{p_i} + \frac{1}{q_i} = 1$, let $p_i(\sigma_i)$ be positive function defined for $\sigma_i \in (0, x_i)$, and define $P_i(t_i) = \int_0^{t_i} p_i(\sigma_i) d\sigma_i$ for $t_i \in (0, x_i)$, where x_i are positive real numbers. Let ϕ_i be real-valued, nonnegative, convex, and sub-multiplicative function defined on $R_+ = [0, \infty)$, $i = 1, 2, \dots, n$. Then

$$(2.13) \quad \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \dots dt_n$$



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 16 of 20

$$\leq \prod_{i=1}^n L(x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left[p_i(t_i) \phi_i \left(\frac{f_i(t_i)}{p_i(t_i)} \right) \right]^{q_i} dt_i \right\}^{\frac{1}{q_i}},$$

where

$$L(x_i, p_i) = \left(\int_0^{x_i} \left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)} \right]^{p_i} dt_i \right)^{\frac{1}{p_i}}.$$

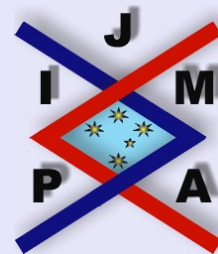
Proof. Applying Jensen's inequality and Hölder's inequality, it is clear to observe that

$$\begin{aligned} \phi_i(F_i(t_i)) &= \phi_i \left(\frac{P_i(t_i) \int_0^{t_i} p_i(\sigma_i) \frac{f_i(\sigma_i)}{p_i(\sigma_i)} d\sigma_i}{\int_0^{t_i} p_i(\sigma_i) d\sigma_i} \right) \\ &\leq \frac{\phi_i(P_i(t_i))}{P_i(t_i)} \int_0^{t_i} p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) d\sigma_i \\ &\leq \frac{\phi_i(P_i(t_i))}{P_i(t_i)} t_i^{\frac{1}{p_i}} \left\{ \int_0^{t_i} \left[p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) \right]^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}. \end{aligned}$$

Using Young's inequality, we obtain that

$$\frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}} \leq \prod_{i=1}^n \frac{\phi_i(P_i(t_i))}{P_i(t_i)} \left\{ \int_0^{t_i} \left[p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) \right]^{q_i} d\sigma_i \right\}^{\frac{1}{q_i}}.$$

Integrating both sides of the above inequality over t_i from 0 to x_i with i running



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

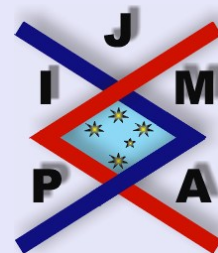
Quit

Page 17 of 20

from i to n , and applying the Hölder inequality, we get

$$\begin{aligned}
 & \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n \phi_i(F_i(t_i))}{\sum_{i=1}^n \frac{t_i}{p_i}} dt_1 dt_2 \cdots dt_n \\
 & \leq \prod_{i=1}^n \left[\int_0^{x_i} \frac{\phi_i(P_i(t_i))}{P_i(t_i)} \left(\int_0^{t_i} \left[p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) \right]^{q_i} d\sigma_i \right)^{\frac{1}{q_i}} dt_i \right] \\
 & \leq \prod_{i=1}^n \left\{ \int_0^{x_i} \left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)} \right]^{p_i} dt_i \right\}^{\frac{1}{p_i}} \left\{ \int_0^{x_i} \left(\int_0^{t_i} \left[p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) \right]^{q_i} d\sigma_i \right) dt_i \right\}^{\frac{1}{q_i}} \\
 & = \prod_{i=1}^n \left\{ \int_0^{x_i} \left[\frac{\phi_i(P_i(t_i))}{P_i(t_i)} \right]^{p_i} dt_i \right\}^{\frac{1}{p_i}} \left\{ \int_0^{x_i} (x_i - t_i) \left[p_i(t_i) \phi_i \left(\frac{f_i(t_i)}{p_i(t_i)} \right) \right]^{q_i} dt_i \right\}^{\frac{1}{q_i}} \\
 & = \prod_{i=1}^n L(x_i, p_i) \left\{ \int_0^{x_i} (x_i - t_i) \left[p_i(t_i) \phi_i \left(\frac{f_i(t_i)}{p_i(t_i)} \right) \right]^{q_i} dt_i \right\}^{\frac{1}{q_i}}
 \end{aligned}$$

This completes the theorem. □



**On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality**

Bing He and Yongjin Li

Title Page

Contents



Go Back

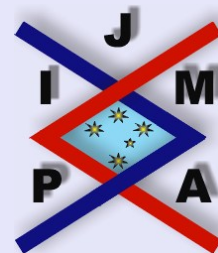
Close

Quit

Page 18 of 20

References

- [1] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.
- [2] B.G. PACHPATTE, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.*, **226**(1) (1998), 166–179.
- [3] G.S. DAVIES AND G.M. PETERSEN, On an inequality of Hardy's, (II), *Quart. J. Math. Oxford Ser.*, **15**(2) (1964), 35–40.
- [4] D.S. MITRINOVIĆ, *Analytic inequalities*. In cooperation with P. M. Vasić. Die Grundlehren der mathematischen Wissenschaften, Band 1965, Springer-Verlag, New York-Berlin, 1970.
- [5] D.S. MITRINOVIĆ AND J.E. PEČARIĆ, On inequalities of Hilbert and Widder, *Proc. Edinburgh Math. Soc.* (2), **34**(3) (1991), 411–414.
- [6] Z. LÜ, Some new inequalities similar to Hilbert-Pachpatte type inequalities, *J. Inequal. Pure Appl. Math.*, **4**(2) (2003), Art. 33. [ONLINE: <http://jipam.vu.edu.au/article.php?sid=271>].
- [7] Y.H. KIM, Some new inverse-type Hilbert-Pachpatte integral inequalities, *Acta Math. Sin.* (Engl. Ser.), **20**(1) (2004), 57–62.
- [8] B. YANG, On an extension of Hilbert's integral inequality with some parameters, *Aust. J. Math. Anal. Appl.*, **1**(1) (2004), Art. 11. [ONLINE: <http://ajmaa.org>].



On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality

Bing He and Yongjin Li

Title Page

Contents



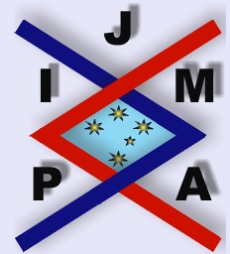
Go Back

Close

Quit

Page 19 of 20

- [9] E.F. BECKENBACH AND R. BELLMAN, *Inequalities*. Second revised printing. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Band 30, Springer-Verlag, New York, Inc. 1965.



**On Several New Inequalities
Close to Hilbert-Pachpatte's
Inequality**

Bing He and Yongjin Li

Title Page

Contents



Go Back

Close

Quit

Page 20 of 20