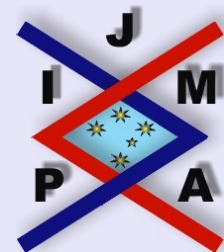


ZHI-GANG WANG, CHUN-YI GAO AND SHAO-MOU YUAN

College of Mathematics and Computing Science
Changsha University of Science and Technology
Changsha, Hunan 410076
People's Republic of China

EMail: zhigwang@163.com



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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

Let $Q(\alpha, \beta, \gamma)$ denote the class of functions of the form $f(z) = z + a_2z^2 + \dots$, which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and satisfy the condition

$$\Re\{\alpha(f(z)/z) + \beta f'(z)\} > \gamma \quad (\alpha, \beta > 0; 0 \leq \gamma < \alpha + \beta \leq 1; z \in \mathcal{U}).$$

The extreme points for this class are provided, the coefficient bounds and radius of univalence for functions belonging to this class are also provided. The results presented here include a number of known results as their special cases.

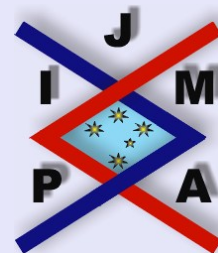
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Contents

1	Introduction	3
2	Extreme Points of the Class $Q(\alpha, \beta, \gamma)$	4
3	Radius of Univalence	8
	References	



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 2 of 10

1. Introduction

Let \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

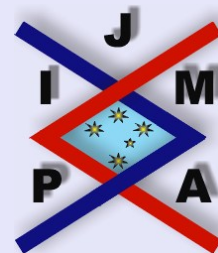
which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. Also let \mathcal{S} denote the familiar subclass of \mathcal{A} consisting of all functions which are univalent in \mathcal{U} .

In the present paper, we consider the following subclass of \mathcal{A} :

$$(1.1) \quad Q(\alpha, \beta, \gamma) = \left\{ f(z) \in \mathcal{A} : \Re \left\{ \alpha \frac{f(z)}{z} + \beta f'(z) \right\} > \gamma \quad (z \in \mathcal{U}) \right\},$$

where $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$.

In some recent papers, Saitoh [2] and Owa [3, 4] discussed the related properties of the class $Q(1 - \beta, \beta, \gamma)$. In the present paper, first we determine the extreme points of the class $Q(\alpha, \beta, \gamma)$, then we find the coefficient bounds and radius of univalence for functions belonging to this class. The results presented here include a number of known results as their special cases.



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 3 of 10

2. Extreme Points of the Class $Q(\alpha, \beta, \gamma)$

First we give the following theorem.

Theorem 2.1. A function $f(z) \in Q(\alpha, \beta, \gamma)$ if and only if $f(z)$ can be expressed as

$$(2.1) \quad f(z) = \frac{1}{\alpha + \beta} \int_{|x|=1} \left[(2\gamma - \alpha - \beta)z + 2(\alpha + \beta - \gamma) \sum_{n=0}^{\infty} \frac{(\alpha + \beta)x^n z^{n+1}}{(n+1)\beta + \alpha} \right] d\mu(x),$$

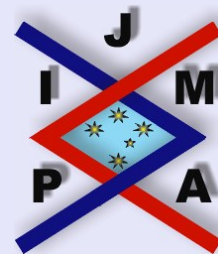
where $\mu(x)$ is the probability measure defined on $X = \{x : |x| = 1\}$. For fixed α, β and γ , $Q(\alpha, \beta, \gamma)$ and the probability measures $\{\mu\}$ defined on X are one-to-one by the expression (2.1).

Proof. By the definition of $Q(\alpha, \beta, \gamma)$, we know $f(z) \in Q(\alpha, \beta, \gamma)$ if and only if

$$\frac{\alpha(f(z)/z) + \beta f'(z) - \gamma}{\alpha + \beta - \gamma} \in \mathcal{P},$$

where \mathcal{P} denotes the normalized well-known class of analytic functions which have positive real part. By the aid of Herglotz expressions of functions in \mathcal{P} , we have

$$\frac{\alpha(f(z)/z) + \beta f'(z) - \gamma}{\alpha + \beta - \gamma} = \int_{|x|=1} \frac{1+xz}{1-xz} d\mu(x),$$



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 4 of 10

or equivalently,

$$\frac{\alpha f(z)}{\beta z} + f'(z) = \frac{1}{\beta} \int_{|x|=1} \frac{\alpha + \beta + (\alpha + \beta - 2\gamma)xz}{1 - xz} d\mu(x).$$

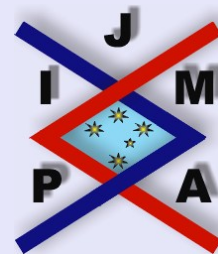
Thus we have

$$\begin{aligned} z^{-\frac{\alpha}{\beta}} \int_0^z \left[\frac{\alpha f(\zeta)}{\beta \zeta} + f'(\zeta) \right] \zeta^{\frac{\alpha}{\beta}} d\zeta \\ = \frac{1}{\beta} \int_{|x|=1} \left[z^{-\frac{\alpha}{\beta}} \int_0^z \frac{\alpha + \beta + (\alpha + \beta - 2\gamma)x\zeta}{1 - x\zeta} \zeta^{\frac{\alpha}{\beta}} d\zeta \right] d\mu(x), \end{aligned}$$

that is,

$$\begin{aligned} f(z) = \frac{1}{\alpha + \beta} \int_{|x|=1} \left[(2\gamma - \alpha - \beta)z \right. \\ \left. + 2(\alpha + \beta - \gamma) \sum_{n=0}^{\infty} \frac{(\alpha + \beta)x^n z^{n+1}}{(n+1)\beta + \alpha} \right] d\mu(x). \end{aligned}$$

This deductive process can be converse, so we have proved the first part of the theorem. We know that both probability measures $\{\mu\}$ and class \mathcal{P} , class \mathcal{P} and $Q(\alpha, \beta, \gamma)$ are one-to-one, so the second part of the theorem is true. This completes the proof of Theorem 2.1. \square



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 5 of 10

Corollary 2.2. *The extreme points of the class $Q(\alpha, \beta, \gamma)$ are*

$$(2.2) \quad f_x(z) = \frac{1}{\alpha + \beta} \left[(2\gamma - \alpha - \beta)z + 2(\alpha + \beta - \gamma) \sum_{n=0}^{\infty} \frac{(\alpha + \beta)x^n z^{n+1}}{(n+1)\beta + \alpha} \right] \quad (|x| = 1).$$

Proof. Using the notation $f_x(z)$, (2.1) can be written as

$$f_\mu(z) = \int_{|x|=1} f_x(z) d\mu(x).$$

By Theorem 2.1, the map $\mu \rightarrow f_\mu$ is one-to-one, so the assertion follows (see [1]). \square

Corollary 2.3. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, then for $n \geq 2$, we have*

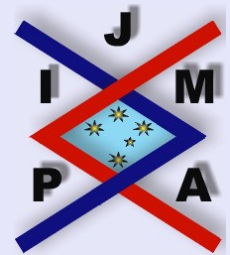
$$|a_n| \leq \frac{2(\alpha + \beta - \gamma)}{n\beta + \alpha}.$$

The results are sharp.

Proof. The coefficient bounds are maximized at an extreme point. Now from (2.2), $f_x(z)$ can be expressed as

$$(2.3) \quad f_x(z) = z + 2(\alpha + \beta - \gamma) \sum_{n=2}^{\infty} \frac{x^{n-1} z^n}{n\beta + \alpha} \quad (|x| = 1),$$

and the result follows. \square



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and Shao-Mou Yuan

Title Page

Contents



Go Back

Close

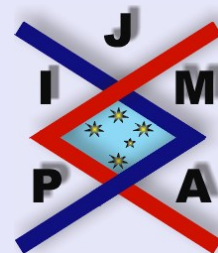
Quit

Page 6 of 10

Corollary 2.4. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q(\alpha, \beta, \gamma)$, then for $|z| = r < 1$, we have

$$|f(z)| \leq r + 2(\alpha + \beta - \gamma) \sum_{n=2}^{\infty} \frac{r^n}{n\beta + \alpha}.$$

This result follows from (2.3).



**On the Univalence of Certain
Analytic Functions**

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 7 of 10

3. Radius of Univalence

In this section, we shall provide the radius of univalence for functions belonging to the class $Q(\alpha, \beta, \gamma)$.

Theorem 3.1. *Let $f(z) \in Q(\alpha, \beta, \gamma)$, then $f(z)$ is univalent in $|z| < R(\alpha, \beta, \gamma)$, where*

$$R(\alpha, \beta, \gamma) = \inf_n \left\{ \frac{n\beta + \alpha}{2n(\alpha + \beta - \gamma)} \right\}^{\frac{1}{n-1}}.$$

This result is sharp.

Proof. It suffices to show that

$$(3.1) \quad |f'(z) - 1| < 1.$$

For the left hand side of (3.1) we have

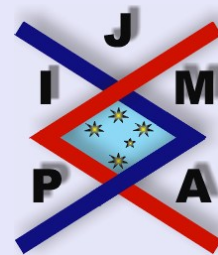
$$\left| \sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n |a_n| |z|^{n-1}.$$

This last expression is less than 1 if

$$|z|^{n-1} < \frac{n\beta + \alpha}{2n(\alpha + \beta - \gamma)}.$$

To show that the bound $R(\alpha, \beta, \gamma)$ is best possible, we consider the function $f(z) \in \mathcal{A}$ defined by

$$f(z) = z - \frac{2(\alpha + \beta - \gamma)}{n\beta + \alpha} z^n.$$



On the Univalence of Certain
Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 8 of 10

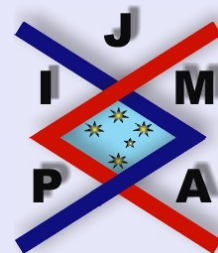
If $\delta > R(\alpha, \beta, \gamma)$, then there exists $n \geq 2$ such that

$$\left\{ \frac{n\beta + \alpha}{2n(\alpha + \beta - \gamma)} \right\}^{\frac{1}{n-1}} < \delta.$$

Since $f'(0) = 1 > 0$ and

$$f'(\delta) = 1 - \frac{2n(\alpha + \beta - \gamma)}{n\beta + \alpha} \delta^{n-1} < 0.$$

Thus, there exists $\delta_0 \in (0, \delta)$ such that $f'(\delta_0) = 0$, which implies that $f(z)$ is not univalent in $|z| < \delta$. This completes the proof of Theorem 3.1. \square



On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

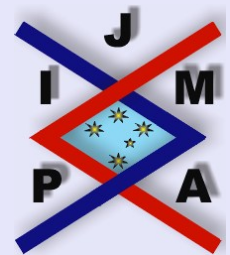
Close

Quit

Page 9 of 10

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On the Univalence of Certain Analytic Functions

Zhi-Gang Wang, Chun-Yi Gao and
Shao-Mou Yuan

Title Page

Contents



Go Back

Close

Quit

Page 10 of 10