Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 162, 2006

A NEW ARRANGEMENT INEQUALITY

MOHAMMAD JAVAHERI

UNIVERSITY OF OREGON
DEPARTMENT OF MATHEMATICS
FENTON HALL, EUGENE, OR 97403
javaheri@uoregon.edu
URL: http://www.uoregon.edu/~javaheri

Received 29 April, 2006; accepted 17 November, 2006
Communicated by G. Bennett

ABSTRACT. In this paper, we discuss the validity of the inequality

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^a x_{i+1}^b \leq \left(\sum_{i=1}^{n} x_i^{(1+a+b)/2} \right)^2 ,$$

where $1, a, b$ are the sides of a triangle and the indices are understood modulo n. We show that, although this inequality does not hold in general, it is true when $n \leq 4$. For general n, we show that any given set of nonnegative real numbers can be arranged as $x_1, x_2, \dots, x_n$ such that the inequality above is valid.

Key words and phrases: Inequality, Arrangement.

2000 *Mathematics Subject Classification.* 26Dxx.

1. MAIN STATEMENTS

Let $a, b, x_1, x_2, \dots, x_n$ be nonnegative real numbers. If $a + b = 1$ then, by the Rearrangement inequality [1], we have

$$\sum_{i=1}^{n} x_i^a x_{i+1}^b \leq \sum_{i=1}^{n} x_i, \tag{1.1}$$

where throughout this paper, the indices are understood to be modulo n. In an attempt to generalize this inequality, we consider the following

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^a x_{i+1}^b \leq \left(\sum_{i=1}^{n} x_i^c \right)^2 , \tag{1.2}$$

ISSN (electronic): 1443-5756

I would like to thank Omar Hosseini for bringing to my attention the case $a = b = 1$ of the inequality (1.2). I would also like to thank Professor G. Bennett for reviewing the article and making useful comments.

126-06

where $c = (a+b+1)/2$. It turns out that if $a+b \neq 1$ then the inequality (1.2) is false for n large enough (cf. Prop. 2.2). However, we show that if

$$b \leq a+1, \qquad a \leq b+1, \qquad 1 \leq a+b, \tag{1.3}$$

then the inequality (1.2) is true in the case of $n = 4$ (cf. Prop. 2.1). Moreover, under the same conditions on a, b as in (1.3), we show that one can always find a permutation μ of $\{1, 2, \ldots, n\}$ such that (cf. Prop. 2.4)

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_{\mu(i)}^a x_{\mu(i+1)}^b \leq \left(\sum_{i=1}^{n} x_i^c\right)^2. \tag{1.4}$$

The conditions in (1.3) cannot be compromised in the sense that if for all nonnegative $x_1, x_2, \ldots, x_n$ there exists a permutation μ such that the conclusion (1.4) holds, then a, b must satisfy (1.3). To see this, let $x_1 = x > 0$ be arbitrary and $x_i = 1, i = 2, \ldots, n$. Then, for any permutation μ, the inequality (1.4) reads the same as:

$$(x+n-1)(x^a + x^b + n - 2) \leq (x^c + n - 1)^2. \tag{1.5}$$

If the above inequality is true for all x and n, by comparing the coefficients of n on both sides of the inequality (1.5), we should have $x^a + x^b + x - 3 \leq 2x^c - 2$. Since $x > 0$ is arbitrary, $1, a, b \leq c$ and conditions (1.3) follow.

The case of $a = b = 1$ of (1.2) is particularly interesting:

$$\sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i x_{i+1} \leq \left(\sum_{i=1}^{n} x_i^{3/2}\right)^2. \tag{1.6}$$

There is a counterexample to (1.6) when $n = 9$, e.g. take

$$\begin{aligned} &x_1 = x_9 = 8.5, \quad x_2 = x_8 = 9, \quad x_3 = x_7 = 10, \\ &x_4 = x_6 = 11.5, \quad x_5 = 12, \end{aligned} \tag{1.7}$$

and subsequently the inequality (1.6) is false for all $n \geq 9$ (cf. prop. (2.2)). Proposition 2.1 shows that the inequality (1.6) is true for $n \leq 4$, and there seems to be a computer-based proof [2] for the cases $n = 5, 6, 7$ which, if true, leaves us with the only remaining case $n = 8$.

2. PROOFS

Applying Jensen's inequality [1, § 3.14] to the concave function $\log x$ gives

$$u^r v^s w^t \leq ru + sv + tw, \tag{2.1}$$

where u, v, w, r, s, t are nonnegative real numbers and $r + s + t = 1$. If, in addition, we have $r, s, t > 0$ then the equality occurs iff $u = v = w$. However, if $t = 0$ and $r, s, w > 0$ then the equality occurs iff $u = v$. We use this inequality in the proof of the proposition below.

Proposition 2.1. *Let $a, b \geq 0$ such that $a+1 \geq b, b+1 \geq a$ and $a+b \geq 1$. Then for all nonnegative real numbers x, y, z, t,*

$$(x+y+z+t)(x^a y^b + y^a z^b + z^a t^b + t^a x^b) \leq (x^c + y^c + z^c + t^c)^2, \tag{2.2}$$

where $c = (a+b+1)/2$. The equality occurs if and only if $\{a, b\} = \{0, 1\}$ or $x = y = z = t$.

Proof. We apply the inequality (2.1) to

$$\begin{aligned} &u = (yz)^c, \quad v = (xz)^c, \quad w = (xy)^c, \\ &r = 1 - \frac{a}{c}, \quad s = 1 - \frac{b}{c}, \quad t = 1 - \frac{1}{c}, \end{aligned} \tag{2.3}$$

and obtain:

$$x^a y^b z \le \left(1-\frac{a}{c}\right)(yz)^c + \left(1-\frac{b}{c}\right)(xz)^c + \left(1-\frac{1}{c}\right)(xy)^c. \tag{2.4}$$

Notice that the assumptions on a, b in the lemma are made exactly so that r, s, t are nonnegative. Similarly, by replacing z with t in (2.4), we have:

$$x^a y^b t \le \left(1-\frac{a}{c}\right)(yt)^c + \left(1-\frac{b}{c}\right)(tx)^c + \left(1-\frac{1}{c}\right)(xy)^c. \tag{2.5}$$

Next, apply (2.1) to

$$u = x^{2c}, \quad v = (xy)^c, \quad w = 1, \quad r = 1-\frac{b}{c}, \quad s = \frac{b}{c}, \quad t = 0, \tag{2.6}$$

and get

$$x^{a+1} y^b \le \left(1-\frac{b}{c}\right) x^{2c} + \frac{b}{c}(xy)^c. \tag{2.7}$$

Similarly, by interchanging a and b, one has

$$x^a y^{b+1} \le \left(1-\frac{a}{c}\right) x^{2c} + \frac{a}{c}(xy)^c. \tag{2.8}$$

Adding the inequalities (2.4), (2.5), (2.7) and (2.8) gives:

$$\begin{aligned} Sx^a y^b \le \frac{1}{c}x^{2c} + \left(4-\frac{3}{c}\right)(xy)^c + \left(1-\frac{a}{c}\right)(yz)^c + \left(1-\frac{b}{c}\right)(tx)^c \\ + \left(1-\frac{a}{c}\right)(yt)^c + \left(1-\frac{b}{c}\right)(xz)^c, \end{aligned} \tag{2.9}$$

where $S = x + y + z + t$. There are three more inequalities of the form above that are obtained by replacing the pair (x, y) by (y, z), (z, t) and (t, x). By adding all four inequalities (or by taking the cyclic sum of (2.9)) we have

$$ST \le \frac{1}{c}\sum x^{2c} + \left(4-\frac{2}{c}\right)(x^c+z^c)(y^c+t^c) + \frac{2}{c}\big\{(xz)^c + (yt)^c\big\}, \tag{2.10}$$

where ST stands for the left hand side of the inequality (2.2). The right hand side of the above inequality is equal to

$$\left(\sum x^c\right)^2 + \left(\frac{1}{c}-1\right)\big\{(x^c+z^c)^2 + (y^c+t^c)^2 - 2(x^c+z^c)(y^c+t^c)\big\}, \tag{2.11}$$

which is less than or equal to $(\sum x^c)^2$, since $c \ge 1$. This concludes the proof of the inequality (2.2).

Next, suppose the equality occurs in (2.2) and so the inequalities (2.4) – (2.8) are all equalities. If $a = 0$ then we have $\sum x \sum x^b = (\sum x^c)^2$ and so, by the equality case of Cauchy-Schwarz, the two vectors (x, y, z, t) and (x^b, y^b, z^b, t^b) have to be proportional. Then either $b = c = 1$ or $x = y = z = t$. Thus suppose $a, b \ne 0$. Since $c = a = b$ is impossible, without loss of generality suppose that $c \ne b$. Since the inequality (2.7) must be an equality, $x^{2c} = x^c y^c$ (cf. the discussion on the equality case of (2.1)). Similarly $y^{2c} = y^c z^c, z^{2c} = z^c t^c$ and $t^{2c} = t^c x^c$. It is then not difficult to see that $x = y = z = t$. □

Let $N(a, b)$ denote the largest integer n for which the inequality (1.2) holds for all nonnegative $x_1, x_2, \dots, x_n$. By the above proposition, we have $N(a, b) \ge 4$.

Proposition 2.2. *Let $a, b \geq 0$ such that $a+b \neq 1$. Then $N(a,b) < \infty$. Moreover, if $n \leq N(a,b)$ then the inequality (1.2) is valid for all nonnegative $x_1, \dots, x_n$.*

Proof. The proof is divided into two parts. First we show that the inequality (1.2) cannot be true for all n. Proof is by contradiction. If $a = b = 0$ then (1.2) is false for $n = 2$ (e.g. take $x_1 = 1, x_2 = 2$). Thus, suppose $a + b > 0$ and that the inequality (1.2) is true for all n. Let f be a non-constant positive continuous function on the interval $I = [0,1]$ such that $f(0) = f(1)$. Let

$$x_i = f\left(\frac{i-1}{n}\right), \quad y_i = (x_i^a x_{i+1}^b)^{1/(a+b)}, \quad i = 1, \dots, n. \tag{2.12}$$

Since y_i is a number between x_i and x_{i+1} (possibly equal to one of them), by the Intermediate-value theorem [3, Th 3.3], there exists $t_i \in I_i$ such that $f(t_i) = y_i$. By the definition of integral we have:

$$\begin{aligned} \int_I f(x)dx \int_I f^{a+b}(x)dx &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^n x_i \sum_{i=1}^n y_i^{a+b} \\ &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^n x_i \sum_{i=1}^n x_i^a x_{i+1}^b \\ &\leq \lim_{n\to\infty} \frac{1}{n^2} \left(\sum_{i=1}^n x_i^c\right)^2 = \left(\int_I f^c(x)dx\right)^2, \end{aligned} \tag{2.13}$$

where we have applied the inequality (1.2) to the x_i's. On the other hand, by the Cauchy-Schwarz inequality for integrals, we have

$$\int_I f(x)dx \int_I f^{a+b}(x)dx \geq \left(\int_I f^{\frac{1}{2}}(x) f^{\frac{a+b}{2}}(x)dx\right)^2 = \left(\int_I f^c(x)dx\right)^2, \tag{2.14}$$

with equality iff f and f^{a+b} are proportional. The statements (2.13) and (2.14) imply that the equality indeed occurs. Since $a + b \neq 1$ and f is not a constant function, the two functions f and f^{a+b} cannot be proportional. This contradiction implies that (1.2) could not be true for all n i.e. $N(a,b) < \infty$.

Next, we show that (1.2) is valid for all $n \leq N$. It is sufficient to show that if the inequality (1.2) is true for all ordered sets of $k+1$ nonnegative real numbers, then it is true for all ordered sets of k nonnegative real numbers.

Let $y_1, \dots, y_k$ be nonnegative real numbers and set

$$S = \sum_{i=1}^k y_i, \qquad A = \sum_{i=1}^k y_i^a y_{i+1}^b, \qquad P = \sum_{i=1}^k y_i^c. \tag{2.15}$$

Without loss of generality we can assume $P = 1$. For each $1 \leq i \leq k$, define an ordered set of $k+1$ nonnegative real numbers by setting:

$$x_j = \begin{cases} y_j & 1 \leq j \leq i+1 \\ y_{j-1} & i+2 \leq j \leq k+1 \end{cases}$$

Applying the inequality (1.2) to $x_1, \dots, x_{k+1}$ gives

$$(S + y_i)(A + y_i^{a+b}) \leq (P + y_i^c)^2 = 1 + y_i^{2c} + 2y_i^c. \tag{2.16}$$

Adding these inequalities for $i = 1, \ldots, k$, yields:

$$kSA + S\sum_i y_i^{a+b} + AS \leq k + 2. \tag{2.17}$$

On the other hand, by the Rearrangement inequality [1] we have

$$\sum_{i=1}^{k} y_i^a y_{i+1}^b \leq \sum_{i=1}^{k} y_i^{a+b}, \tag{2.18}$$

and the lemma follows by putting together the inequalities (2.17) and (2.18). □

The inequality (1.1) translates to $N(a, b) = \infty$ when $a+b = 1$. We expect that $N(a, b) \to \infty$ as $a + b \to 1$. The following proposition supports this conjecture. We define

$$A_n(a, b) = \sup\left\{ \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^a x_{i+1}^b - \left(\sum_{i=1}^{n} x_i^c\right)^2 \,\middle|\, \max_{1\leq i\leq n} x_i = 1 \right\}. \tag{2.19}$$

This number roughly measures the validity of the inequality (1.2). Also let

$$\sigma_t = \frac{1}{n}\sum_{i=1}^{n} x_i^t. \tag{2.20}$$

By the Hölder inequality [1], if $\alpha, \beta > 0$ and $\alpha + \beta = 1$ then for any $s, t > 0$ we have:

$$\sigma_s^\alpha \sigma_t^\beta \geq \sigma_{\alpha s+\beta t}. \tag{2.21}$$

Proposition 2.3. *$N(u, u)$ is a non-increasing function of $u \geq 1/2$. Moreover, for all n and $a, b \geq 0$*

$$\lim_{a+b\to 1} A_n(a, b) = 0. \tag{2.22}$$

Proof. Suppose $u > v > 1/2$. We show that $N(u, u) \leq N(v, v)$. Without loss of generality we can assume:

$$u - v < \frac{1}{4}. \tag{2.23}$$

By the definition of $N = N(v, v)$, there must exist $N + 1$ nonnegative integers $x_1, \ldots, x_{N+1}$ such that the inequality (1.2) is false and so

$$\sum_{i=1}^{N+1} x_i \sum_{i=1}^{N+1} x_i^v x_{i+1}^v > \left(\sum_{i=1}^{N+1} x_i^{v+1/2}\right)^2. \tag{2.24}$$

We show that the nonnegative numbers $y_i = x_i^{u/v}$, $i = 1, \ldots, N + 1$ give a counterexample to (1.2) when $a = b = u$. In light of (2.24), one just needs to show

$$\left(\sum_{i=1}^{N+1} x_i^{u+1/2v}\right)^2 \Bigg/ \sum_{i=1}^{N+1} x_i^{u/v} \geq \left(\sum_{i=1}^{N+1} x_i^{u+1/2}\right)^2 \Bigg/ \sum_{i=1}^{N+1} x_i. \tag{2.25}$$

To prove this, first let

$$\alpha = \frac{u + 1/(2v) - u/v}{u + 1/(2v) - 1}, \quad \beta = \frac{u/v - 1}{u + 1/(2v) - 1}, \tag{2.26}$$
$$s = 1, \quad t = u + \frac{1}{2v}.$$

The numbers above are simply chosen such that $\alpha+\beta=1$ and $\alpha s+\beta t=u/v$. We briefly check that $\alpha,\beta>0$. The denominator of fractions above is positive, since $u+1/(2v)\geq(v+1/v)/2\geq 1$. This implies $\beta>0$. Now the positivity of $\alpha>0$ is equivalent to $u(1-v)<1/2$. If $v\geq 1$ then $u(1-v)\leq 0<1/2$. So suppose $v\leq 1$. By using (2.23), we have:

$$u(1-v)\leq\left(v+\frac{1}{4}\right)(1-v)=-v^2+\frac{3}{4}v+\frac{1}{4}<\frac{1}{2}, \tag{2.27}$$

for all $v\geq 0$. Now we can safely plug α,β,s,t in (2.21) and get

$$\sigma_1^{\alpha}\sigma_{u+1/2v}^{\beta}\geq\sigma_{u/v}. \tag{2.28}$$

Next, let $\alpha'=(1-\alpha)/2$ and $\beta'=1-\beta/2$. Since $\alpha'+\beta'=1$ and $\alpha',\beta'>0$, we can use Hölder's inequality (2.21) with α',β' instead of α and β (and the same s,t as before) and get (this time $\alpha's+\beta't=u+1/2$):

$$\sigma_1^{(1-\alpha)/2}\sigma_{1+1/2v}^{1-\beta/2}\leq\sigma_{u+1/2}. \tag{2.29}$$

Now we square the above inequality and multiply it with (2.28) to obtain:

$$\sigma_1\sigma_{1+1/2v}^2\leq\sigma_{u/v}\sigma_{u+1/2}^2, \tag{2.30}$$

which is equivalent to the inequality (2.25). So far we have shown the existence of a counterexample to (1.2) for $a=b=u$ when $n=N+1$. Then Prop. 2.2 gives $N(u,u)\leq N=N(v,v)$ and this concludes the proof of the monotonicity of N.

It remains to prove that $A_n(a,b)$ converges to 0 as $a+b\to 1$. To the contrary, assume there exists $\epsilon>0$ and a sequence (a_j,b_j) such that $A_n(a_j,b_j)>\epsilon$ and $a_j+b_j\to 1$. Then by definition, for each j, there exists an n-tuple $X_j=(x_{1j},\ldots,x_{nj})$ such that $\max x_{ij}=1$ and

$$\sum_{i=1}^n x_{ij}\sum_{i=1}^n x_{ij}^{a_j}x_{i+1j}^{b_j}-\left(\sum_{i=1}^n x_{ij}^{c_j}\right)^2\geq\frac{\epsilon}{2}, \tag{2.31}$$

where $c_j=(a_j+b_j+1)/2$. Since X_j is a bounded sequence, it follows that, along a subsequence j_k, the X_{j_k}'s converge to some $X=(x_1,\ldots,x_n)$. On the other hand, along a subsequence of j_k (denoted again by j_k), $a_{j_k}\to a$ and $b_{j_k}\to b$ for some $a,b\geq 0$. Since $a_j+b_j\to 1$, we have $a+b=1$. By taking the limits of the inequality (2.31) along this subsequence, we should have

$$\sum_{i=1}^n x_i\sum_{i=1}^n x_i^a x_{i+1}^b-\left(\sum_{i=1}^n x_i\right)^2\geq\frac{\epsilon}{2}>0, \tag{2.32}$$

which contradicts the inequality (1.1). This contradiction establishes the equation (2.22). □

The next proposition shows that the inequality (1.2) holds if one mixes up the order of the x_i's. The proof is simple and makes use of the monotonicity of $(\sigma_t)^{1/t}$ where σ_t is defined by the equation (2.20). It is well-known that $(\sigma_t)^{1/t}$ is a non-decreasing function of t [1, Th. 16].

Proposition 2.4. *Let a,b,c be as in Proposition 2.1. Then for any given set of n nonnegative real numbers there exists an arrangement of them as $x_1,\ldots,x_n$ such that the inequality (1.2) holds.*

Proof. Equivalently, we show that if $x_1,x_2,\ldots,x_n$ are nonnegative then there exists a permutation μ of the set $\{1,2,\ldots,n\}$ such that the inequality (1.4) holds. Let

$$S=\sum_{i=1}^n x_i,\qquad T=\sum_{i=1}^n\sum_{j\neq i}x_i^a x_j^b. \tag{2.33}$$

Then $ST = n\sigma_1(n^2\sigma_a\sigma_b - n\sigma_{a+b}) = n^3\sigma_1\sigma_a\sigma_b - n^2\sigma_1\sigma_{a+b}$. Now by the Cauchy-Schwarz inequality [4], $\sigma_c^2 \leq \sigma_1\sigma_{a+b}$. On the other hand by the monotonicity of $\sigma_t^{1/t}$, we have $\sigma_1 \leq \sigma_c^{1/c}, \sigma_a \leq \sigma_c^{a/c}, \sigma_b \leq \sigma_c^{b/c}$, and so $\sigma_1\sigma_a\sigma_b \leq \sigma_c^2$. It follows from these inequalities that

$$ST \leq n^2(n-1)\sigma_c^2. \tag{2.34}$$

Now for a permutation μ of $1, 2, \ldots, n$, let:

$$A_\mu = \sum_{i=1}^{n} x_{\mu(i)}^a x_{\mu(i+1)}^b. \tag{2.35}$$

We would like to show that $SA_\mu \leq (n\sigma_c)^2$ for some permutation μ. It is sufficient to show that the average of SA_μ over all permutations μ is less than or equal to $(n\sigma_c)^2$. To show this, observe that the average of SA_μ is equal to $ST/(n-1)$ and so the claim follows from the inequality (2.34). □

The symmetric group S_n acts on $\mathbb{R}^n$ in the usual way, namely for $\mu \in S_n$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$ let

$$\mu \cdot (x_1, \ldots, x_n) = (x_{\mu(1)}, \ldots, x_{\mu(n)}). \tag{2.36}$$

Let R be a region in $\mathbb{R}^n$ that is invariant under the action of permutations (i.e. $\mu \cdot R \subseteq R$ for all μ). define:

$$\lambda(R) = \left\{ (x_1, \ldots, x_n) \in R \,\middle|\, \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^a x_{i+1}^b \leq \left(\sum_{i=1}^{n} x_i^c \right)^2 \right\}. \tag{2.37}$$

By Proposition 2.4:

$$R \subseteq \bigcup_{\mu \in S_n} \mu \cdot \lambda(R). \tag{2.38}$$

In particular, by taking the Lebesgue measure of the sides of the inclusion above, we get

$$\operatorname{vol} \lambda(R) \geq \frac{\operatorname{vol} R}{n!}. \tag{2.39}$$

We prove a better lower bound for $\operatorname{vol} \lambda(R)$ when n is a prime number (similar but weaker results can be proved in general).

Proposition 2.5. *Let a, b be as in Proposition 2.1 and n be a prime number. Let $R \subseteq \mathbb{R}_+^n$ be a Lebesgue-measurable bounded set that is invariant under the action of permutations. Let $\lambda(R)$ denote the set of all $(x_1, \ldots, x_n) \in R$ for which the inequality (1.2) holds. Then*

$$\operatorname{vol} \lambda(R) \geq \frac{\operatorname{vol} R}{n-1}. \tag{2.40}$$

Proof. For $m \in \{1, 2, \ldots, n-1\}$ let $\mu_m \in S_n$ and denote the permutation

$$\mu_m(i) = mi, \tag{2.41}$$

where all the numbers are understood to be modulo n (in particular $\mu_m(n) = n$ for all m). Now recall the definition of A_μ from the equation (2.35) and observe that:

$$\begin{aligned} \sum_{m=1}^{n-1} A_{\mu_m} &= \sum_{m=1}^{n-1} \sum_{i=1}^{n} x_{mi}^a x_{mi+m}^b = \sum_{m=1}^{n-1} \sum_{j=1}^{n} x_j^a x_{j+m}^b \\ &= \sum_{j=1}^{n} x_j^a \sum_{m=1}^{n-1} x_{j+m}^b = \sum_{j=1}^{n} x_j^a \sum_{i \neq j} x_i^b. \end{aligned} \tag{2.42}$$

Then, the same argument in the proof of Prop. 2.4 implies that, for some $m \in \{1, \ldots, n-1\}$, we have $A_{\mu_m} \leq (n\sigma_c)^2$. We conclude that

$$R \subseteq \bigcup_{m=1}^{n-1} \mu_m \cdot \lambda(R), \tag{2.43}$$

which in turn implies the inequality (2.40). □

REFERENCES

[1] G.H. HARDY, J.E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, 2nd ed. (1988).

[2] O. HOSSEINI, *Private Communication*, Fall 2006.

[3] M.H. PROTTER AND C.B. Jr MORREY, *A First Course in Real Analysis*, Springer (1997).

[4] M.J. STEELE, *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Cambridge University Press (2004).