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ON TWO PROBLEMS POSED BY KENNETH STOLARSKY

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ABSTRACT. Solutions of two slightly more general problems than those posed by Kenneth B. Stolarsky in [10] are presented. The latter deal with a shape preserving approximation, in the uniform norm, of two functions $(1/x)\log\cosh x$ and $(1/x)\log(\sinh x/x)$, $x \geq 0$, by ratios of exponomials. The main mathematical tools employed include Gini means and the Stolarski means.

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1. INTRODUCTION

The purpose of this note is to present solutions of two problems posed by Professor Kenneth B. Stolarsky in [10, p. 817]. They are formulated as follows:

> "Call (as is sometimes done) a polynomial in $x, \exp(c_1 x), \ldots, \exp(c_n x)$ an *exponomial*. Alternatively, an exponomial is a solution of the constant coefficient linear differential equation. Is there a sequence of functions $f_n(x)$, $n = 1, 2, 3, \ldots$, each a ratio of exponomials and each increasing from 0 to 1 as x increases from 0 to ∞, such that

(1) $f_n''(x) \leq 0$ for all $x \geq 0$,
(2) either $f_n(x) \leq f_m(x)$ for all $x \geq 0$ or $f_m(x) \leq f_n(x)$ for all $x \geq 0$,
(3) assertion (2) remains valid if $f_m(x)$ is replaced by $(1/x)\log\cosh x$ (or by $(1/x)\log(\sinh x/x)$), and
(4) in some neighborhood of the graph $y = (1/x)\log\cosh x$ (or of $(1/x)\log(\sinh x/x)$) the graphs of the $f_n(x)$ are dense with respect to the uniform (supremum) norm?"

Let us note that both functions $(1/x)\log\cosh x$ and $(1/x)\log(\sinh x/x)$ are concave functions on $\mathbb{R}_+ -$ the nonnegative semi-axis and they increase from zero to one as x increases from

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zero to infinity. Thus these problems can be regarded as the approximation problems, in the uniform norm, with the shape constraints imposed on the approximating functions. In what follows we will refer to these problems as the first Stolarsky problem and the second Stolarsky problem, respectively.

This paper is organized as follows. In Section 2 we recall definitions and basic properties of two families of the bivariate means. They are employed in solutions of two slightly more general problems than those mentioned earlier in this section. The main results are contained in Sections 3 and 4.

2. GINI MEANS AND STOLARSKY MEANS

Let $p, q \in \mathbb{R}$ and let $a, b \in \mathbb{R}_>$ – the positive semi-axis. The Gini mean $G_{p,q}(a,b)$ of order (p,q) of a and b is defined as

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{a^p + b^p}{a^q + b^q}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\dfrac{a^p \log a + b^p \log b}{a^p + b^p}\right), & p = q \end{cases} \tag{2.1}$$

(see [1]). For later use, let us record some properties of this two-parameter family of means:

(P1) $G_{p,q}$ increases with an increase in either p and q (see [7]).
(P2) If $p > 0$ and $q > 0$, then $G_{p,q}$ is log-concave in both p and q. If $p < 0$ and $q < 0$, then $G_{p,q}$ is log-convex in both p and q (see [6]).
(P3) If $p \neq q$, then

$$\log G_{p,q}(a,b) = \frac{1}{p-q}\int_q^p \log J_t(a,b)\,dt,$$

where

$$J_t(a,b) = G_{t,t}(a,b) \qquad (t \in \mathbb{R}). \tag{2.2}$$

Let us note that $G_{p,0}(a,b) = A_p(a,b)$, $p \neq 0$, where

$$A_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}} \tag{2.3}$$

is the Hölder mean (power mean) of order p of a and b.

A second family of means used here has been introduced by K.B. Stolarsky in [9]. Throughout the sequel we will denote them by $D_{p,q}(a,b)$ where again $p, q \in \mathbb{R}$ and $a, b \in \mathbb{R}_>$. For $a \neq b$ they are defined as

$$D_{p,q}(a,b) = \begin{cases} \left(\dfrac{q}{p}\dfrac{a^p - b^p}{a^q - b^q}\right)^{\frac{1}{p-q}}, & pq(p-q) \neq 0 \\ \exp\left(-\dfrac{1}{p} + \dfrac{a^p \log a - b^p \log b}{a^p - b^p}\right), & p = q \neq 0 \\ \left[\dfrac{a^p - b^p}{p(\log a - \log b)}\right]^{\frac{1}{p}}, & p \neq 0,\ q = 0 \\ \sqrt{ab}, & p = q = 0 \end{cases} \tag{2.4}$$

and $D_{p,q}(a,a) = a$.

They have the monotonicity and concavity (convexity) properties analogous to those listed in (P1) and (P2) (see [3], [8], [9]). Also, if $p \neq q$, then

$$\log D_{p,q}(a,b) = \frac{1}{p-q} \int_q^p \log I_t(a,b)\, dt, \tag{2.5}$$

where

$$I_t(a,b) = D_{t,t}(a,b) \tag{2.6}$$

is the identric mean of order t ($t \in \mathbb{R}$) of a and b (see [9]). Let us note that $A_p(a,b) = D_{2p,p}(a,b)$ and $L_p(a,b) = D_{p,0}(a,b)$ is the logarithmic mean of order p ($p \in \mathbb{R}$) of a and b.

Comparison results for the Gini means and Stolarsky means are discussed in a recent paper [5].

3. A Generalization of the First Stolarsky Problem and Its Solution

In this section we deal with a generalization of the first Stolarsky problem. Its solution is also included here.

For $(p,q) \in \mathbb{R}^2_+$ let

$$f(p,q) = \begin{cases} \dfrac{1}{p-q} \log \left(\dfrac{\cosh p}{\cosh q} \right), & p \neq q \\ \tanh p, & p = q. \end{cases} \tag{3.1}$$

A function to be approximated in the first Stolarsky problem is equal to $f(x,0)$ (see Section 1). Making use of (2.1) we see that

$$f(p,q) = \log G_{p,q}(e, e^{-1}). \tag{3.2}$$

It follows from (P1)–(P3), (3.1), and (3.2) that

(i) $0 \le f(p,q) < 1$,

(ii) $f(p,q)$ increases along any ray $d = \lambda(\alpha, \beta)$, where $\lambda \ge 0$, $(\alpha, \beta) \in \mathbb{R}^2_+$ ($\alpha + \beta > 0$),

(iii) function $f(p,q)$ is concave in both variables p and q, and

(iv) $$f(p,q) = \frac{1}{p-q} \int_q^p \tanh t\, dt$$

provided $p \neq q$.

For later use we define functions

$$g_n(p,q) = \frac{1}{n} \sum_{k=1}^{n} \tanh(\alpha_k p + \beta_k q) \quad (n = 1, 2, \ldots), \tag{3.3}$$

where

$$\alpha_k = \frac{2k-1}{2n}, \qquad \beta_k = 1 - \alpha_k \quad (1 \le k \le n). \tag{3.4}$$

One can easily verify that the function $g_n(p,q)$ is a ratio of two exponomials, $0 \le g_n(p,q) < 1$, and $g_n(p,q)$ increases along any ray $d = \lambda(\alpha, \beta)$, where λ, α, and β are the same as in (ii). Moreover, $g_n(p,q)$ is a concave function on $\mathbb{R}^2_+$. In order to prove the last statement, let

$$\phi_k(p,q) = \tanh t,$$

where $t = \alpha_k p + \beta_k q$ ($1 \le k \le n$). An easy computation shows that the Hessian $H\phi_k$ of ϕ_k is equal to

$$H\phi_k = -2 \tanh(t) \operatorname{sech}^2(t) \begin{bmatrix} \alpha_k^2 & \alpha_k \beta_k \\ \alpha_k \beta_k & \beta_k^2 \end{bmatrix}.$$

The eigenvalues λ_1 and λ_2 of $H\phi_k$ satisfy $\lambda_2 < \lambda_1 = 0$. This in turn implies that the function $\phi_k(p,q)$ is concave on $\mathbb{R}^2_+$. The same conclusion is valid for the function $g_n(p,q)$ because of (3.3).

We are in a position to prove the main result of this section.

Theorem 3.1. *Let* $0 \le p, q < \infty$ *and let*

$$f_m(p,q) = g_{2^m}(p,q) \quad (m = 0, 1, \dots). \tag{3.5}$$

Then

(a) $f_m(p,q)$ *is a ratio of two exponomials.*
(b) $0 \le f_m(p,q) < 1$.
(c) $f_m(p,q)$ *increases along any ray* $d = \lambda(\alpha,\beta)$*, where* λ*,* α*, and* β *are the same as in (ii).*
(d) $f_m(p,q)$ *is a concave function on* $\mathbb{R}^2_+$*.*
(e) $\lim_{m\to\infty} \|f - f_m\|_\infty = 0$*, where* $\|\cdot\|_\infty$ *stands for the uniform norm on* $\mathbb{R}^2_+$*.*
(f) *The inequalities* $f(p,q) \le f_{m+1}(p,q) \le f_m(p,q)$ *are valid for all* $m = 0, 1, \dots$*.*

Proof. Statements (a)–(d) follow from the properties of the function $g_n(p,q)$, established earlier in this section, and from (3.5). For the proof of (e) it suffices to show that

$$\lim_{n\to\infty} \|f - g_n\|_\infty = 0. \tag{3.6}$$

To this aim we recall the Composite Midpoint Rule (see e.g., [2])

$$\int_0^1 h(t)\,dt = \frac{1}{n}\sum_{k=1}^{n} h(\alpha_k) + \frac{1}{24n^2}h''(\xi) \quad (n \ge 1), \tag{3.7}$$

where the numbers α_k are defined in (3.4) and $0 < \xi < 1$. Application of (3.7) to (iv), with $h(t) = \tanh t$, gives

$$\begin{aligned} f(p,q) &= \int_0^1 \tanh(up + (1-u)q)\,du \\ &= g_n(p,q) - \frac{1}{12}\left(\frac{p-q}{n}\right)^2 \frac{\tanh(\xi p + (1-\xi)q)}{\cosh^2(\xi p + (1-\xi)q)}. \end{aligned}$$

This in conjunction with the inequality $0 \le \tanh x/\cosh^2 x \le 1/2$ ($x \ge 0$) gives

$$0 \le g_n(p,q) - f(p,q) \le \frac{1}{24n^2}(p-q)^2 \tag{3.8}$$

($n = 1, 2, \dots$). The convergence results (3.6) and (e) now follow. Moreover, the first inequality in (3.8) give, together with (3.5), the first inequality in (f). To complete the proof of (f) we use (3.5), (3.3), and (3.4) to obtain

$$f_{m+1}(p,q) = \frac{1}{2^{m+1}}\sum_{k=1}^{2^{m+1}} \tanh(\gamma_k p + \delta_k q), \tag{3.9}$$

where

$$\gamma_k = \frac{2k-1}{2^{m+2}} \quad \text{and} \quad \delta_k = 1 - \gamma_k, \quad 1 \le k \le 2^{m+1}.$$

Since $\tanh t$ is concave for $t \geq 0$, (3.9) gives

$$\begin{aligned} f_{m+1}(p,q) &= \frac{1}{2^m} \sum_{k=1,3,\ldots,2^{m+1}-1} \frac{1}{2}\left[\tanh(\gamma_k p + \delta_k q) + \tanh(\gamma_{k+1} p + \delta_{k+1} q)\right] \\ &\leq \frac{1}{2^m} \sum_{k=1,3,\ldots,2^{m+1}-1} \tanh\left(\frac{\gamma_k + \gamma_{k+1}}{2} p + \frac{\delta_k + \delta_{k+1}}{2} q\right) \\ &= \frac{1}{2^m} \sum_{k=1}^{2^m} \tanh(\alpha_k p + \beta_k q) = f_m(p,q), \end{aligned}$$

where now

$$\alpha_k = \frac{2k-1}{2^{m+1}}, \quad \beta_k = 1 - \alpha_k, \quad 1 \leq k \leq 2^m.$$

The proof is complete. □

4. A Generalization and a Solution of the Second Stolarsky Problem

This section is devoted to the discussion of a generalization of the second Stolarsky problem. In what follows we will use the same symbols for both, a function to be approximated and the approximating functions, as those employed in Section 3.

For $(p,q) \in \mathbb{R}_+^2$, let

$$f(p,q) = \begin{cases} \dfrac{1}{p-q} \log\left(\dfrac{q}{p}\dfrac{\sinh p}{\sinh q}\right), & pq(p-q) \neq 0; \\ \coth p - \dfrac{1}{p}, & p = q \neq 0; \\ \dfrac{1}{p} \log\left(\dfrac{\sinh p}{p}\right), & p \neq 0,\ q = 0; \\ 0, & p = q = 0. \end{cases} \tag{4.1}$$

Stolarsky's function of his second problem is a particular case of $f(p,q)$, namely $f(x,0)$. Making use of (2.4) we obtain

$$f(p,q) = \log D_{p,q}(e, e^{-1}). \tag{4.2}$$

Function $f(p,q)$ defined in (4.1) possesses the same properties as those listed in (i)–(iii) (see Section 3). A counterpart of the integral formula in (iv) reads as follows

$$f(p,q) = \frac{1}{p-q} \int_q^p \left(\coth t - \frac{1}{t}\right) dt \qquad (p \neq q). \tag{4.3}$$

This is an immediate consequence of (2.5), (2.6), (4.2), and (4.1).

For $n = 1, 2, \ldots$, we define

$$g_n(p,q) = \frac{1}{n} \sum_{k=1}^{n} \left[\coth(\alpha_k p + \beta_k q) - \frac{1}{\alpha_k p + \beta_k q}\right], \tag{4.4}$$

where α_k and β_k are defined in (3.4). Again, one can easily verify that the function $g_n(p,q)$ has the same monotonicity and concavity properties as its counterpart defined in (3.3). Also, we define functions $f_m(p,q)$ as

$$f_m(p,q) = g_{2^m}(p,q) \quad (m = 0, 1, \ldots).$$

Since the main result of this section can be formulated in exactly the same way as Theorem 3.1, we omit further details with the exception of the proof of uniform convergence of the functions $f_m(p,q)$ to the function $f(p,q)$.

Application of the Composite Midpoint Rule (3.7) to the integral on the right side of (4.3) gives

$$f(p,q) = g_n(p,q) - \frac{1}{12}\left(\frac{p-q}{n}\right)^2 \phi(t), \tag{4.5}$$

where

$$\phi(t) = \frac{1}{t^3} - \frac{\coth t}{\sinh^2 t}, \quad t = \xi p + (1-\xi)q, \quad 0 < \xi < 1.$$

Function $\phi(u)$ is nonnegative for $u \geq 0$. This follows from the Lazarević inequality $\cosh u \leq (\sinh u/u)^3$ (see, e.g., [4, p. 270]). Moreover,

$$\phi(u) = \frac{1}{15}u - \frac{4}{189}u^3 + \frac{1}{225}u^5 - \cdots \leq \frac{1}{15}u,$$

where the last inequality is valid provided $u \geq 0$. This in conjunction with (4.5) gives

$$0 \leq g_n(p,q) - f(p,q) \leq \frac{1}{180n^2}(p-q)^2 \max(p,q) \qquad (n = 1, 2, \ldots).$$

Since p and q are nonnegative finite numbers, we conclude that

$$\lim_{n\to\infty} \|f - g_n\|_\infty = 0.$$

The uniform convergence of the sequence $\{f_m(p,q)\}_0^\infty$ to the function $f(p,q)$ now follows.

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