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ON AN INEQUALITY OF V. CSISZÁR AND T.F. MÓRI FOR CONCAVE FUNCTIONS OF TWO VARIABLES

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Abstract. V. Csiszár and T.F. Móri gave an extension of Diaz-Metcalf's inequality for concave functions. In this paper, we show its restatement. As its applications we first give a reverse inequality of Hölder's inequality. Next we consider two variable versions of Hadamard, Petrović and Giaccardi inequalities.

Key words and phrases: Diaz-Metcalf inequality, Hölder's inequality, Hadamard's inequality, Petrović's inequality, Giaccardi's inequality.

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1. Introduction

In this paper, let (X, Y) be a random vector with $P[(X, Y) \in D] = 1$ where $D := [a, A] \times [b, B]$ ($0 \leq a < A$ and $0 \leq b < B$). Let $E[X]$ be the expectation of a random variable X with respect to P. For a function $\phi : D \to \mathbb{R}$, we put

$$\Delta\phi = \Delta\phi(a, b, A, B) := \phi(a, b) - \phi(a, B) - \phi(A, b) + \phi(A, B).$$

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In [1], V. Csiszár and T.F. Móri showed the following theorem as an extension of Diaz-Metcalf's inequality [2].

Theorem A. *Let $\phi : D \to \mathbb{R}$ be a concave function.*
We use the following notations:

$$\lambda_1 = \lambda_4 := \frac{\phi(A,b)-\phi(a,b)}{A-a}, \quad \mu_1 = \mu_3 := \frac{\phi(a,B)-\phi(a,b)}{B-b},$$
$$\lambda_2 = \lambda_3 := \frac{\phi(A,B)-\phi(a,B)}{A-a}, \quad \mu_2 = \mu_4 := \frac{\phi(A,B)-\phi(A,b)}{B-b}, \quad \textit{and}$$
$$\nu_1 := \frac{AB-ab}{(A-a)(B-b)}\phi(a,b) - \frac{b}{B-b}\phi(a,B) - \frac{a}{A-a}\phi(A,b),$$
$$\nu_2 := \frac{A}{A-a}\phi(a,B) + \frac{B}{B-b}\phi(A,b) - \frac{AB-ab}{(A-a)(B-b)}\phi(A,B),$$
$$\nu_3 := \frac{B}{B-b}\phi(a,b) - \frac{a}{A-a}\phi(A,B) + \frac{aB-Ab}{(A-a)(B-b)}\phi(a,B),$$
$$\nu_4 := \frac{A}{A-a}\phi(a,b) - \frac{b}{B-b}\phi(A,B) - \frac{aB-Ab}{(A-a)(B-b)}\phi(A,b).$$

$a)$ *Suppose that $\Delta\phi \geq 0$.*
$a-(\mathrm{i})$ *If $(B-b)E[X]+(A-a)E[Y] \leq AB-ab$, then*

$$\lambda_1 E[X] + \mu_1 E[Y] + \nu_1 \leq E[\phi(X,Y)] \ (\leq \phi(E[X],E[Y]))\,.$$

$a-(\mathrm{ii})$ *If $(B-b)E[X]+(A-a)E[Y] \geq AB-ab$, then*

$$\lambda_2 E[X] + \mu_2 E[Y] + \nu_2 \leq E[\phi(X,Y)] \ (\leq \phi(E[X],E[Y]))\,.$$

$b)$ *Suppose that $\Delta\phi \leq 0$*
$b-(\mathrm{iii})$ *If $(B-b)E[X]+(A-a)E[Y] \leq aB-Ab$, then*

$$\lambda_3 E[X] + \mu_3 E[Y] + \nu_3 \leq E[\phi(X,Y)] \ (\leq \phi(E[X],E[Y]))\,.$$

$b-(\mathrm{iv})$ *If $(B-b)E[X]+(A-a)E[Y] \geq aB-Ab$, then*

$$\lambda_4 E[X] + \mu_4 E[Y] + \nu_4 \leq E[\phi(X,Y)] \ (\leq \phi(E[X],E[Y]))\,.$$

Let us note that Theorem A can be given in the following form:

Theorem 1.1. *Suppose that $\phi : D \to \mathbb{R}$ is a concave function.*
$a)$ *If $\Delta\phi \geq 0$, then*

$$\max_{k=1,2}\{\lambda_k E[X] + \mu_k E[Y] + \nu_k\} \leq E[\phi(X,Y)](\leq \phi(E[X],E[Y]), \tag{1.1}$$

where λ_k, μ_k and ν_k ($k=1,2$) are defined in Theorem A.
$b)$ *If $\Delta\phi \leq 0$,then*

$$\max_{k=3,4}\{\lambda_k E[X] + \mu_k E[Y] + \nu_k\} \leq E[\phi(X,Y)](\leq \phi(E[X],E[Y]), \tag{1.2}$$

where λ_k, μ_k and ν_k ($k=3,4$) are defined in Theorem A.

Remark 1.2. The inequality $E[\phi(X,Y)] \leq \phi(E[X],E[Y])$ is Jensen's inequality. So the inequalities in Theorem A represent reverse inequalities of it.

In this note, we shall give some applications of these results.

2. REVERSE HÖLDER'S INEQUALITY

Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\phi(x, y) := x^{\frac{1}{p}} y^{\frac{1}{q}}$ is a concave function on $(0, \infty) \times (0, \infty)$. For $0 < a < A$ and $0 < b < B$, $\Delta\phi$ is represented as follows:

$$\Delta\phi = a^{\frac{1}{p}} b^{\frac{1}{q}} - a^{\frac{1}{p}} B^{\frac{1}{q}} - A^{\frac{1}{p}} b^{\frac{1}{q}} + A^{\frac{1}{p}} B^{\frac{1}{q}} = \left(A^{\frac{1}{p}} - a^{\frac{1}{p}}\right)\left(B^{\frac{1}{q}} - b^{\frac{1}{q}}\right) \ (> 0).$$

Moreover, putting $A = B = 1$, and replacing X, Y, a and b by X^p, Y^q, α^p and β^q, respectively, in Theorem A, we have the following result:

Theorem 2.1. *Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 < \alpha \le X \le 1$ and $0 < \beta \le Y \le 1$.*
(i) *If $(1-\beta^q)E[X^p] + (1-\alpha^p)E[Y^q] \le 1 - \alpha^p\beta^q$, then*

$$\frac{\beta(1-\alpha)}{1-\alpha^p}E[X^p] + \frac{\alpha(1-\beta)}{1-\beta^q}E[Y^q] + \frac{\alpha\beta(1-\alpha^{p-1}-\beta^{q-1}+\alpha^{p-1}\beta^q+\alpha^p\beta^{q-1}-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)} \le E[XY]. \tag{2.1}$$

(ii) *If $(1-\beta^q)E[X^p] + (1-\alpha^p)E[Y^q] \ge 1 - \alpha^p\beta^q$, then*

$$\frac{1-\alpha}{1-\alpha^p}E[X^p] + \frac{1-\beta}{1-\beta^q}E[Y^q] - \frac{1-\alpha-\beta+\alpha\beta^q+\alpha^p\beta-\alpha^p\beta^q}{(1-\alpha^p)(1-\beta^q)} \le E[XY]. \tag{2.2}$$

By Theorem 2.1 we have the following inequality related to Hölder's inequality:

Theorem 2.2. *Let $p, q > 1$ be real numbers with $\frac{1}{p} + \frac{1}{q} = 1$. If $0 < \alpha \le X \le 1$ and $0 < \beta \le Y \le 1$, then*

$$\begin{aligned} &p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}} E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}} \\ &\le (\beta - \alpha\beta^q)E[X^p] + (\alpha - \alpha^p\beta)E[Y^q] \\ &\le (1 - \alpha^p\beta^q)E[XY]. \end{aligned} \tag{2.3}$$

Proof. We have by Young's inequality

$$\begin{aligned} &(\beta - \alpha\beta^q)E[X^p] + (\alpha - \alpha^p\beta)E[Y^q] \\ &= \frac{1}{p} \cdot p(\beta - \alpha\beta^q)E[X^p] + \frac{1}{q} \cdot q(\alpha - \alpha^p\beta)E[Y^q] \\ &\ge \{p(\beta - \alpha\beta^q)E[X^p]\}^{\frac{1}{p}} \{q(\alpha - \alpha^p\beta)E[Y^q]\}^{\frac{1}{q}} \\ &= p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}} E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}}. \end{aligned}$$

Hence the first inequality holds. Next, we see that

$$-\gamma_1 := \frac{\alpha\beta(1-\alpha^{p-1}-\beta^{q-1}+\alpha^{p-1}\beta^q+\alpha^p\beta^{q-1}-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)} \ge 0 \tag{2.4}$$

and

$$\gamma_2 := \frac{1-\alpha-\beta+\alpha\beta^q+\alpha^p\beta-\alpha^p\beta^q}{(1-\alpha^p)(1-\beta^q)} \ge 0. \tag{2.5}$$

Indeed, we have $(1-\alpha^p)(1-\beta^q) > 0$ and moreover by Young's inequality

$$\begin{aligned} 1 &- \alpha^{p-1} - \beta^{q-1} + \alpha^{p-1}\beta^q + \alpha^p\beta^{q-1} - \alpha^p\beta^q \\ &= 1 - \alpha^p\beta^q - \alpha^{p-1}(1-\beta^q) - \beta^{q-1}(1-\alpha^p) \\ &\geq 1 - \alpha^p\beta^q - \left(\frac{1}{p} + \frac{1}{q}\alpha^p\right)(1-\beta^q) - \left(\frac{1}{q} + \frac{1}{p}\beta^q\right)(1-\alpha^p) = 0 \end{aligned}$$

and

$$\begin{aligned} 1 &- \alpha - \beta + \alpha\beta^q + \alpha^p\beta - \alpha^p\beta^q \\ &= 1 - \alpha^p\beta^q - \alpha(1-\beta^q) - \beta(1-\alpha^p) \\ &\geq 1 - \alpha^p\beta^q - \left(\frac{1}{q} + \frac{1}{p}\alpha^p\right)(1-\beta^q) - \left(\frac{1}{p} + \frac{1}{q}\beta^q\right)(1-\alpha^p) = 0. \end{aligned}$$

Multiplying both sides of (2.1) by γ_2 and those of (2.2) by $-\gamma_1$, respectively, and taking the sum of the two inequalities, we have

$$\begin{vmatrix} \frac{\beta(1-\alpha)}{1-\alpha^p} & \gamma_1 \\ \frac{1-\alpha}{1-\alpha^p} & \gamma_2 \end{vmatrix} E[X^p] + \begin{vmatrix} \frac{\alpha(1-\beta)}{1-\beta^q} & \gamma_1 \\ \frac{1-\beta}{1-\beta^q} & \gamma_2 \end{vmatrix} E[Y^q] \leq (\gamma_2 - \gamma_1)E[XY].$$

Here we note that from (2.4) and (2.5),

$$\begin{vmatrix} \frac{\beta(1-\alpha)}{1-\alpha^p} & \gamma_1 \\ \frac{1-\alpha}{1-\alpha^p} & \gamma_2 \end{vmatrix} = \frac{\beta(1-\alpha)(1-\beta)(1-\alpha\beta^{q-1})}{(1-\alpha^p)(1-\beta^q)},$$

$$\begin{vmatrix} \frac{\alpha(1-\beta)}{1-\beta^q} & \gamma_1 \\ \frac{1-\beta}{1-\beta^q} & \gamma_2 \end{vmatrix} = \frac{\alpha(1-\alpha)(1-\beta)(1-\alpha^{p-1}\beta)}{(1-\alpha^p)(1-\beta^q)}$$

and

$$\gamma_2 - \gamma_1 = \frac{(1-\alpha)(1-\beta)(1-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)}.$$

Hence we have

$$\begin{aligned} \frac{\beta(1-\alpha)(1-\beta)(1-\alpha\beta^{q-1})}{(1-\alpha^p)(1-\beta^q)}E[X^p] &+ \frac{\alpha(1-\alpha)(1-\beta)(1-\alpha^{p-1}\beta)}{(1-\alpha^p)(1-\beta^q)}E[Y^q] \\ &\leq \frac{(1-\alpha)(1-\beta)(1-\alpha^p\beta^q)}{(1-\alpha^p)(1-\beta^q)}E[XY] \end{aligned}$$

and so the second inequality of (2.3) holds. □

The second inequality is given in [5, p.124]. In (2.3), the first and the third terms yield the following Gheorghiu inequality [4, p.184], [5, p.124]:

Theorem B. *Let* $p, q > 1$ *be real numbers with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $0 < \alpha \leq X \leq 1$ *and* $0 < \beta \leq Y \leq 1$, *then*

$$E[X^p]^{\frac{1}{p}} E[Y^q]^{\frac{1}{q}} \leq \frac{1-\alpha^p\beta^q}{p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}}} E[XY]. \tag{2.6}$$

We see that (2.3) is a kind of a refinement of (2.6). Theorem B gives us the next estimation.

Corollary 2.3. *Let $X = \{a_i\}$ and $Y = \{b_j\}$ be independent discrete random variables with distributions $P(X = a_i) = w_i$ and $P(Y = b_j) = z_j$. Suppose $0 < \alpha \le X \le 1$ and $0 < \beta \le Y \le 1$. $E[X^p], E[Y^q]$ and $E[XY]$ are given by $\sum_{i=1}^n w_i a_i^p, \sum_{i=1}^n z_j b_j^q$ and $\sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j$, respectively. Then we have inequalities*

$$\sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j \le \left(\sum_{i=1}^n w_i a_i^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^n z_j b_j^q\right)^{\frac{1}{q}}$$
$$\le \frac{1-\alpha^p\beta^q}{p^{\frac{1}{p}} q^{\frac{1}{q}} (\beta - \alpha\beta^q)^{\frac{1}{p}} (\alpha - \alpha^p\beta)^{\frac{1}{q}}} \sum_{i=1}^n \sum_{j=1}^n w_i z_j a_i b_j.$$

3. HADAMARD'S INEQUALITY

The following well-known inequality is due to Hadamard [5, p.11]: For a concave function $f : [a, b] \to \mathbb{R}$,

$$\frac{f(a)+f(b)}{2} \le \frac{1}{b-a}\int_a^b f(t)dt \le f\left(\frac{a+b}{2}\right). \tag{3.1}$$

Moreover, the following is an extension of the weighted version of Hadamard's inequality by Fejér ([3], [6, p.138]): Let g be a positive integrable function on $[a, b]$ with $g(a + t) = g(b - t)$ for $0 \le t \le \frac{1}{2}(a - b)$. Then

$$\frac{f(a)+f(b)}{2}\int_a^b g(t)dt \le \int_a^b f(t)g(t)dt \le f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt. \tag{3.2}$$

Here we give an analogous result for a function of two variables.

Theorem 3.1. *Let X and Y be independent random variables such that*

$$E[X] = \frac{a+A}{2} \quad \text{and} \quad E[Y] = \frac{b+B}{2} \tag{3.3}$$

for $0 < a \le X \le A$ and $0 < b \le Y \le B$. If $\phi : D \to \mathbb{R}$ is a concave function, then

$$\min\left\{\frac{\phi(A,b)+\phi(a,B)}{2}, \frac{\phi(a,b)+\phi(A,B)}{2}\right\} \le E[\phi(X,Y)] \le \phi\left(\frac{a+A}{2}, \frac{b+B}{2}\right). \tag{3.4}$$

Proof. We only have to prove the case $\Delta\phi \ge 0$. Then with same notations as in Theorem A we have

$$\lambda_1 E[X] + \mu_1 E[Y] + \nu_1 = \lambda_2 E[X] + \mu_2 E[Y] + \nu_2 = \frac{\phi(A,b)+\phi(a,B)}{2}$$

by (3.3). Since $\Delta\phi \ge 0$, it is the same as the first expression in (3.4). Similarly calculation for $\Delta\phi \le 0$ proves that the desired inequality (3.4) also holds. $\square$

We can obtain the following result as an extension of Hadamard's inequality (3.1) from Theorem 3.1 by letting X and Y be independent, uniformly distrbuted radom variables on the intervals $[a, A]$ and $[b, B]$, respectively:

Corollary 3.2. *Let $0 < a < A$ and $0 < b < B$. If ϕ is a concave function, then*

$$\min\left\{\frac{\phi(A,b)+\phi(a,B)}{2}, \frac{\phi(a,b)+\phi(A,B)}{2}\right\} \le \frac{1}{(A-a)(B-b)}\int_a^A\int_b^B \phi(t,s)\,dsdt \le \phi\left(\frac{a+A}{2}, \frac{b+B}{2}\right).$$

By Theorem 3.1, we have the following analogue of (3.2) for a function of two variables:

Corollary 3.3. *Let $w : D \to \mathbb{R}$ be a nonnegative integrable function such that $w(s,t) = u(s)v(t)$ where $u : [a,A] \to \mathbb{R}$ is an integrable function with $u(s) = u(a+A-s)$, $\int_a^A u(s)ds = 1$ and $v : [b,B] \to \mathbb{R}$ is an integrable function such that $\int_b^B v(t)dt = 1$, $v(t) = v(b+B-t)$. If ϕ is a concave function, then*

$$\min\left\{\frac{\phi(A,b)+\phi(a,B)}{2}, \frac{\phi(a,b)+\phi(A,B)}{2}\right\} \le \int_a^A\int_b^B w(s,t)\phi(s,t)\,dsdt \le \phi\left(\frac{a+A}{2}, \frac{b+B}{2}\right).$$

4. PETROVIĆ'S INEQUALITY

The following is called Petrović's inequality for a concave function $f : [0,c] \to \mathbb{R}$:

$$f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(x_i) + (1-P_n)\,f(0),$$

where $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{p} = (p_1,\ldots,p_n)$ are n-tuples of nonnegative real numbers such that $\sum_{i=1}^n p_i x_i \ge x_k$ for $k = 1,\ldots,n$, $\sum_{i=1}^n p_i x_i \in [0,c]$ and $P_n := \sum_{i=1}^n p_i$ (see [5, p.11] and [6]).

We give an analogous result for a function of two variables.

Theorem 4.1. *Let $\mathbf{p} = (p_1,\ldots,p_n)$ and $\mathbf{q} = (q_1,\ldots,q_n)$ be n-tuples of nonnegative real numbers and put $P_n := \sum_{i=1}^n p_i\ (>0)$ and $Q_n := \sum_{j=1}^n q_j\ (>0)$. Suppose that $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ are n-tuples of nonnegative real numbers with $0 \le x_k \le \sum_{i=1}^n p_i x_i \le c$ and $0 \le y_k \le \sum_{j=1}^n q_j y_j \le d$ for $k = 1,2,\ldots,n$. Let $\phi : [0,c]\times[0,d] \to \mathbb{R}$ be a concave function.*
$a)$ *Suppose*

$$\phi(0,0) + \phi\left(\sum_{i=1}^n p_i x_i, \sum_{j=1}^n q_j y_j\right) \ge \phi\left(\sum_{i=1}^n p_i x_i, 0\right) + \phi\left(0, \sum_{j=1}^n q_j y_j\right).$$

$a-(\mathrm{i})$ *If $\frac{1}{P_n} + \frac{1}{Q_n} \le 1$, then*

$$\frac{1}{P_n}\phi\left(\sum_{i=1}^n p_i x_i, 0\right) + \frac{1}{Q_n}\phi\left(0, \sum_{j=1}^n q_j y_j\right) + \left(1 - \frac{1}{P_n} - \frac{1}{Q_n}\right)\phi(0,0) \le \frac{1}{P_n Q_n}\sum_{i=1}^n\sum_{j=1}^n p_i q_j \phi(x_i, y_j). \tag{4.1}$$

$a-(\mathrm{ii})$ *If* $\frac{1}{P_n}+\frac{1}{Q_n}\geq 1$*, then*

$$\left(\frac{1}{P_n}+\frac{1}{Q_n}-1\right)\phi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)+\left(1-\frac{1}{Q_n}\right)\phi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\left(1-\frac{1}{P_n}\right)\phi\left(0,\sum_{j=1}^{n}q_jy_j\right)\leq \frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\phi(x_i,y_j).$$

$b)$ *Suppose*

$$\phi(0,0)+\phi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)\leq\phi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\phi\left(0,\sum_{j=1}^{n}q_jy_j\right).$$

$b-(\mathrm{iii})$ *If* $P_n\geq Q_n$*, then*

$$\frac{1}{P_n}\phi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)+\left(\frac{1}{Q_n}-\frac{1}{P_n}\right)\phi\left(0,\sum_{j=1}^{n}q_jy_j\right)+\left(1-\frac{1}{Q_n}\right)\phi(0,0)\leq \frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\phi(x_i,y_j).$$

$b-(\mathrm{iv})$ *If* $Q_n\geq P_n$*, then*

$$\frac{1}{Q_n}\phi\left(\sum_{i=1}^{n}p_ix_i,\sum_{j=1}^{n}q_jy_j\right)-\left(\frac{1}{Q_n}-\frac{1}{P_n}\right)\phi\left(\sum_{i=1}^{n}p_ix_i,0\right)+\left(1-\frac{1}{P_n}\right)\phi(0,0)\leq \frac{1}{P_nQ_n}\sum_{i=1}^{n}\sum_{j=1}^{n}p_iq_j\phi(x_i,y_j).$$

Proof. We put $a=b=0$, $A=\sum_{i=1}^{n}p_ix_i$ and $B=\sum_{j=1}^{n}q_jy_j$ in Theorem A. Let $X=\{a_i\}$ and $Y=\{b_j\}$ be independent discrete random variables with distributions $P(X=x_i)=\frac{p_i}{P_n}$ and $P(Y=y_j)=\frac{q_i}{Q_n}$, $1\leq i\leq n$, respectively. So we have the desired inequalities. □

Specially, if $p_i=q_j=1$ $(i,j=1,\dots,n)$ in Theorem 4.1, then we have the following:

Corollary 4.2. *Suppose that* $\mathbf{x}=(x_1,\dots,x_n)$ *and* $\mathbf{y}=(y_1,\dots,y_n)$ *are* n*-tuples of nonnegative real numbers for* $n\geq 2$ *with* $\sum_{i=1}^{n}x_i\in[0,c]$ *and* $\sum_{i=1}^{n}y_i\in[0,d]$*. If* $\phi:[0,c]\times[0,d]\to\mathbb{R}$ *is a concave function, then*

$$\phi\left(\sum_{i=1}^{n}x_i,0\right)+\phi\left(0,\sum_{j=1}^{n}y_j\right)+(n-2)\,\phi(0,0)\leq\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\phi(x_i,y_j). \tag{4.2}$$

5. GIACCARDI'S INEQUALITY

In 1955, Giaccardi (cf. [5, p.11]) proved the following inequality for a convex function $f : [a, A] \to \mathbb{R}$,

$$\sum_{i=1}^{n} p_i f(x_i) \leq C \cdot f\left(\sum_{i=1}^{n} p_i x_i\right) + D \cdot (P_n - 1) \cdot f(x_0),$$

where

$$C = \frac{\sum_{i=1}^{n} p_i(x_i - x_0)}{\sum_{i=1}^{n} p_i x_i - x_0} \quad \text{and} \quad D = \frac{\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i - x_0}$$

for a nonnegative n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$ with $P_n := \sum_{i=1}^{n} p_i$ and a real $(n+1)$-tuple $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ such that for $k = 0, 1, \ldots, n$

$$a \leq x_i \leq A, \quad (x_k - x_0)\left(\sum_{i=1}^{n} p_i x_i - x_0\right) \geq 0,$$
$$a < \sum_{i=1}^{n} p_i x_i < A \quad \text{and} \quad \sum_{i=1}^{n} p_i x_i \neq x_0.$$

In this section, we discuss a generalization of Giaccardi's inequality to a function of two variables under similar conditions. Let $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ and $\mathbf{y} = (y_0, y_1, \ldots, y_n)$ be nonnegative $(n+1)$-tuples, and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be nonnegative n-tuples with

$$x_0 \leq x_k \leq \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad y_0 \leq y_k \leq \sum_{j=1}^{n} q_j y_j \quad \text{for } k = 1, \ldots, n. \tag{5.1}$$

We use the following notations:

$$P_n := \sum_{i=1}^{n} p_i \ (\geq 0), \qquad Q_n := \sum_{j=1}^{n} q_j \ (\geq 0),$$
$$K(X) := \frac{\sum_{i=1}^{n} p_i x_i - P_n x_0}{\sum_{i=1}^{n} p_i x_i - x_0}, \qquad K(Y) := \frac{\sum_{j=1}^{n} q_j y_j - Q_n y_0}{\sum_{j=1}^{n} q_j y_j - y_0},$$
$$L(X) := \frac{(P_n - 1)\sum_{i=1}^{n} p_i x_i}{\sum_{i=1}^{n} p_i x_i - x_0}, \qquad L(Y) := \frac{(Q_n - 1)\sum_{j=1}^{n} q_j y_j}{\sum_{j=1}^{n} q_j y_j - y_0},$$

$$M(X,Y) := \left\{(P_n Q_n - P_n - Q_n)\sum_{i=1}^{n} p_i x_i \sum_{j=1}^{n} q_j y_j + Q_n y_0 \sum_{i=1}^{n} p_i x_i + P_n x_0 \sum_{j=1}^{n} q_j y_j - P_n Q_n x_0 y_0\right\} \times \frac{1}{\left(\sum_{i=1}^{n} p_i x_i - x_0\right)\left(\sum_{j=1}^{n} q_j y_j - y_0\right)}$$

and

$$N(X,Y) := \frac{(P_n - Q_n)\sum_{i=1}^{n} p_i x_i \sum_{j=1}^{n} q_j y_j - (P_n - 1)Q_n \sum_{i=1}^{n} p_i x_i y_0 + P_n(Q_n - 1)x_0 \sum_{j=1}^{n} q_j y_j}{\left(\sum_{i=1}^{n} p_i x_i - x_0\right)\left(\sum_{j=1}^{n} q_j y_j - y_0\right)}.$$

Then we have the following theorem:

Theorem 5.1. *Let* $\phi : [x_0, \sum_{i=1}^{n} p_i x_i] \times [y_0, \sum_{j=1}^{n} q_j y_j] \to \mathbb{R}$ *be a concave function.*
$a)$ *If*

$$\phi(x_0, y_0) + \phi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right) \geq \phi\left(x_0, \sum_{j=1}^{n} q_j y_j\right) + \phi\left(\sum_{i=1}^{n} p_i x_i, y_0\right),$$

then

$$\max\left\{Q_n K(X)\phi\left(\sum_{i=1}^{n} p_i x_i, y_0\right) + P_n K(Y)\phi\left(x_0, \sum_{j=1}^{n} q_j y_j\right) + M(X,Y)\phi(x_0, y_0)\right.,$$
$$\left. P_n L(Y)\phi\left(\sum_{i=1}^{n} p_i x_i, y_0\right) + Q_n L(X)\phi\left(x_0, \sum_{j=1}^{n} q_j y_j\right) - M(X,Y)\phi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right)\right\}$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \phi(x_i, y_j).$$

$b)$ *If*

$$\phi(x_0, y_0) + \phi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right) \leq \phi\left(x_0, \sum_{j=1}^{n} q_j y_j\right) + \phi\left(\sum_{i=1}^{n} p_i x_i, y_0\right),$$

then

$$\max\left\{Q_n K(X)\phi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right) + P_n L(Y)\phi\left(x_0, y_0\right) + N(X,Y)\phi\left(x_0, \sum_{j=1}^{n} q_j y_j\right)\right.,$$
$$\left. P_n K(Y)\phi\left(\sum_{i=1}^{n} p_i x_i, \sum_{j=1}^{n} q_j y_j\right) + Q_n L(X)\phi\left(x_0, y_0\right) - N(X,Y)\phi\left(\sum_{i=1}^{n} p_i x_i, y_0\right)\right\}$$
$$\leq \sum_{i=1}^{n}\sum_{j=1}^{n} p_i q_j \phi(x_i, y_j).$$

Proof. Let X and Y be as they were in the proof of Theorem 4.1, and put $a = x_0$, $A = \sum_{i=1}^{n} p_i x_i$, $b = y_0$ and $B = \sum_{j=1}^{n} q_j y_j$, and use Theorem A. Then we have the desired inequalities of this theorem. □

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