



ON INEQUALITIES FOR HYPERGEOMETRIC ANALOGUES OF THE ARITHMETIC-GEOMETRIC MEAN

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Roger W. Barnard and
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[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 1 of 12

[Go Back](#)

[Full Screen](#)

[Close](#)

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Abstract: In this note, we present sharp inequalities relating hypergeometric analogues of the arithmetic-geometric mean discussed in [5] and the power mean. The main result generalizes the corresponding sharp inequality for the arithmetic-geometric mean established in [10].

Contents

1	Introduction	3
2	Main Results	6
3	Applications	9



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vol. 8, iss. 3, art. 65, 2007

[Title Page](#)

[Contents](#)



Page 2 of 12

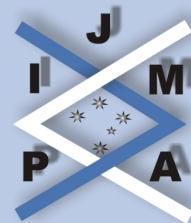
[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**
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mathematics

issn: 1443-5756



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 12

Go Back

Full Screen

Close

1. Introduction

In 1799, Gauss made a remarkable discovery (see equation (1.2) below) regarding the closed form of the compound mean created by iteratively applying the arithmetic mean \mathcal{A}_1 and geometric mean \mathcal{A}_0 , which are special cases of

$$\mathcal{A}_\lambda(a, b) \equiv \left(\frac{a^\lambda + b^\lambda}{2} \right)^{\frac{1}{\lambda}} \quad (\lambda \neq 0),$$

with $\mathcal{A}_0(a, b) \equiv \sqrt{ab}$ for $a, b > 0$. A standard argument reveals that the power mean \mathcal{A}_λ is an increasing function of its order λ . In particular, the arithmetic and geometric means satisfy the well-known inequality $\mathcal{A}_0(a, b) \leq \mathcal{A}_1(a, b)$. From this it can be shown that the recursively defined sequences given by $a_{n+1} = \mathcal{A}_1(a_n, b_n)$, $b_{n+1} = \mathcal{A}_0(a_n, b_n)$ (with $b_0 = b < a = a_0$) satisfy

$$\mathcal{A}_0(a, b) \leq b_n < b_{n+1} < a_{n+1} < a_n \leq \mathcal{A}_1(a, b) \quad \text{for all } n \in \mathbb{N}.$$

Thus $\{a_n\}$, $\{b_n\}$ are bounded and monotone sequences satisfying

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \mathcal{A}_1(a_n, b_n) = \lim_{n \rightarrow \infty} \mathcal{A}_0(a_n, b_n) = \lim_{n \rightarrow \infty} b_{n+1},$$

by continuity and the fact that these means are strict (i.e. $\mathcal{A}_\lambda(a, b) = a$ iff $a = b$). It is this common limit which is used to define the compound mean $\mathcal{A}_1 \otimes \mathcal{A}_0(a, b) \equiv \lim_{n \rightarrow \infty} a_n$, commonly referred to as the *arithmetic-geometric mean* $\mathcal{AG} \equiv \mathcal{A}_1 \otimes \mathcal{A}_0$. Moreover, the convergence is quadratic for this particular compound iteration. For more on the historical development of \mathcal{AG} , the article [1] by Almkvist and Berndt and the text *Pi and the AGM* by Borwein and Borwein [3] are lively and informative sources.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 12

Go Back

Full Screen

Close

By construction, $\mathcal{A}_0(a, b) < \mathcal{AG}(a, b) < \mathcal{A}_1(a, b)$ for $a > b > 0$. However, \mathcal{A}_1 is not the best possible power mean upper bound for \mathcal{AG} . For example, since

$$a_2 = \frac{\frac{a+b}{2} + \sqrt{ab}}{2} = \left(\frac{\sqrt{a} + \sqrt{b}}{2} \right)^2 = \mathcal{A}_{1/2}(a, b),$$

it follows that

$$\mathcal{A}_0(a, b) < \mathcal{AG}(a, b) < \mathcal{A}_{1/2}(a, b) \quad \text{for all } a > b > 0.$$

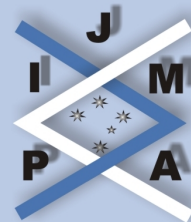
Vamanamurthy and Vuorinen [10] showed that the order $1/2$ is *sharp*. As a result

$$(1.1) \quad \mathcal{A}_\lambda(a, b) < \mathcal{AG}(a, b) < \mathcal{A}_\mu(a, b) \quad \text{for all } a > b > 0$$

if and only if $\lambda \leq 0$ and $\mu \geq 1/2$. The aim of this note is to discuss sharp inequalities that parallel (1.1) for hypergeometric analogues of the arithmetic-geometric mean introduced in [5] and described below.

A review of the above iterative process leading to \mathcal{AG} reveals that any two continuous strict means \mathcal{M}, \mathcal{N} can be used to construct a compound mean, provided \mathcal{M} is comparable to \mathcal{N} (i.e. $\mathcal{M}(a, b) \geq \mathcal{N}(a, b)$ for $a \geq b > 0$). Moreover, $\mathcal{M} \otimes \mathcal{N}$ inherits standard mean properties such as homogeneity (i.e. $\mathcal{M}(sa, sb) = s\mathcal{M}(a, b)$ for $s > 0$) when possessed by both \mathcal{M} and \mathcal{N} (see [3, p. 244]). While the definition of the compound mean as the limit of an iterative process is pleasingly simple, it is natural to pursue a closed-form expression to facilitate further analysis. Gauss engaged in this pursuit for \mathcal{AG} and his discovery yields the following elegant identity (see [3, 9]):

$$(1.2) \quad \mathcal{AG}(1, r) = \frac{1}{{}_2F_1(1/2, 1/2; 1; 1 - r^2)},$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 12

Go Back

Full Screen

Close

where ${}_2F_1$ is the Gaussian hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; z) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n, \quad |z| < 1,$$

and $(\alpha)_n \equiv \Gamma(\alpha + n)/\Gamma(\alpha) = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ for $n \in \mathbb{N}$, $(\alpha)_0 \equiv 1$.

Using modular forms, Borwein et al. (see [5]) constructed quadratically convergent compound means that can be expressed in closed form as

$$(1.3) \quad \mathcal{M} \otimes \mathcal{N}(1, r) = \frac{1}{{}_2F_1(1/2 - s, 1/2 + s; 1; 1 - r^p)^q}.$$

Motivated by a comparison with (1.2), compound means satisfying (1.3) are described in [5] as *hypergeometric analogues* of \mathcal{AG} . Sharp inequalities similar to (1.1) for these “close relatives” of \mathcal{AG} can be obtained by applying the following theorem from [8] involving the *hypergeometric mean* ${}_2F_1(-a, b; c; r)^{1/a}$ (discussed by Carlson in [6]) and the weighted power mean given by

$$\mathcal{A}_\lambda(\omega; a, b) \equiv [\omega a^\lambda + (1 - \omega) b^\lambda]^{1/\lambda} \quad (\lambda \neq 0)$$

and $\mathcal{A}_0(\omega; a, b) \equiv a^\omega b^{1-\omega}$, with weights $\omega, 1 - \omega > 0$.

Theorem 1.1 ([8]). *Suppose $1 \geq a$, $b > 0$ and $c > \max\{-a, b\}$. If $c \geq \max\{1 - 2a, 2b\}$, then*

$$\mathcal{A}_\lambda(1 - b/c; 1, 1 - r) \leq {}_2F_1(-a, b; c; r)^{1/a}, \quad \forall r \in (0, 1)$$

if and only if $\lambda \leq \frac{a+c}{1+c}$. If $c \leq \min\{1 - 2a, 2b\}$, then

$$\mathcal{A}_\lambda(1 - b/c; 1, 1 - r) \geq {}_2F_1(-a, b; c; r)^{1/a}, \quad \forall r \in (0, 1)$$

if and only if $\lambda \geq \frac{a+c}{1+c}$.



2. Main Results

The principal contribution of this note is the observation that Theorem 1.1 can be used to obtain sharp upper bounds for the hypergeometric analogues of \mathcal{AG} . We also note that the corresponding lower bounds can be verified directly using elementary series techniques presented here (or as a corollary to more involved developments as in [7]). Simultaneous sharp bounds of this type are of independent interest.

Proposition 2.1. *Suppose $0 < \alpha \leq 1/2$. Then for all $r \in (0, 1)$*

$$(2.1) \quad \mathcal{A}_\lambda(\alpha; 1, r^\alpha) < \frac{1}{{}_2F_1(\alpha, 1 - \alpha; 1; 1 - r)} < \mathcal{A}_\mu(\alpha; 1, r^\alpha)$$

if and only if $\lambda \leq 0$ and $\mu \geq (1 - \alpha)/(2\alpha)$.

Proof. By the monotonicity of $\lambda \mapsto \mathcal{A}_\lambda$, it suffices to verify the first inequality in (2.1) for the elementary case that $\lambda = 0$. It follows easily by induction that $\frac{(\alpha(1-\alpha))_n}{n!} \geq \frac{(\alpha)_n(1-\alpha)_n}{n!n!}$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} (1 - r)^{-\alpha(1-\alpha)} &= \sum_{n=0}^{\infty} \frac{(\alpha(1 - \alpha))_n}{n!} r^n \\ &> \sum_{n=0}^{\infty} \frac{(\alpha)_n(1 - \alpha)_n}{n!n!} r^n = {}_2F_1(\alpha, 1 - \alpha; 1; r). \end{aligned}$$

This implies

$$\mathcal{A}_0(\alpha; 1, (1 - r)^\alpha) = (1 - r)^{\alpha(1-\alpha)} < {}_2F_1(\alpha, 1 - \alpha; 1; r)^{-1}.$$

The replacement of r by $(1 - r)$ completes a proof of the established first inequality in (2.1) for $\lambda \leq 0$. Sharpness follows from the observation that if $\lambda > 0$, then

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 6 of 12

Go Back

Full Screen

Close

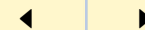
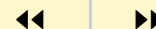
journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756



Title Page

Contents



Page 7 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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$\mathcal{A}_\lambda(\alpha; 1, 0) > 0$ while ${}_2F_1(\alpha, 1 - \alpha; 1; r)^{-1} \rightarrow 0$ as $r \rightarrow 1^-$ (see [9, p. 111]). Thus, for $\lambda > 0$ and r sufficiently close to and less than 1, it follows that

$$\mathcal{A}_\lambda(\alpha; 1, (1 - r)^\alpha) - {}_2F_1(1/2, 1/2; 1; r)^{-1} > 0.$$

That is, $\lambda \leq 0$ is necessary and sufficient for the first inequality in (2.1).

The proof of the second inequality is not as obvious. From Theorem 1.1, if $\alpha = -a > 0$, $\beta = 1 - \alpha > 0$ and $\max\{\alpha, \beta\} < \gamma \leq \min\{1 + 2\alpha, 2\beta\}$, then for all $r \in (0, 1)$

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; r)^{-1/\alpha} &\leq \left[\left(1 - \frac{\beta}{\gamma}\right) + \frac{\beta}{\gamma}(1 - r)^\sigma \right]^{\frac{1}{\sigma}} \\ &= \mathcal{A}_\sigma \left(1 - \frac{\beta}{\gamma}; 1, 1 - r\right) \end{aligned}$$

for the sharp order $\sigma = (\gamma - \alpha)/(1 + \gamma)$. (By the proof of Theorem 1.1 in [8], the above inequality is strict unless $\gamma = 1 + 2\alpha = 2\beta$). The conditions for strict inequality are met for $0 < \alpha \leq 1/2$, $\beta = 1 - \alpha$, $\gamma = 1$. Thus

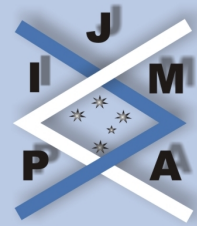
$${}_2F_1(\alpha, 1 - \alpha; 1; 1 - r)^{-1} < \mathcal{A}_\sigma(\alpha; 1, r)^\alpha \quad \text{for all } r \in (0, 1),$$

if and only if $\sigma \geq (1 - \alpha)/2$. Noting that $\mathcal{A}_\sigma(\omega; 1, r)^\alpha = \mathcal{A}_{\sigma/\alpha}(\omega; 1, r^\alpha)$, we obtain the second inequality in (2.1) for $\mu = \sigma/\alpha$. \square

Corollary 2.2. *Suppose $0 < \alpha \leq 1/2$ and $p > 0$. Then for all $r \in (0, 1)$*

$$(2.2) \quad \mathcal{A}_\lambda(\alpha; 1, r) < \frac{1}{{}_2F_1(\alpha, 1 - \alpha; 1; 1 - r^p)^{\frac{1}{\alpha p}}} < \mathcal{A}_\mu(\alpha; 1, r)$$

if and only if $\lambda \leq 0$ and $\mu \geq p(1 - \alpha)/2$.



Proof. Proposition 2.1 implies that for all $r \in (0, 1)$ and $q > 0$

$$\mathcal{A}_{\hat{\lambda}}(\alpha; 1, r^{p\alpha})^q < \frac{1}{{}_2F_1(\alpha, 1 - \alpha; 1; 1 - r^p)^q} < \mathcal{A}_{\hat{\mu}}(\alpha; 1, r^{p\alpha})^q$$

if and only if $\hat{\lambda} \leq 0$ and $\hat{\mu} \geq (1 - \alpha)/(2\alpha)$. Since

$$\mathcal{A}_{\hat{\mu}}(\alpha; 1, r^{p\alpha})^q = \mathcal{A}_{\hat{\mu}/q}(\alpha; 1, r^{pq\alpha}),$$

the result follows by setting $\lambda = \hat{\lambda}/q$ and $\mu = \hat{\mu}/q$ for $pq\alpha = 1$. □

It is interesting to note that properties of the important class of *zero-balanced* hypergeometric functions of the form ${}_2F_1(a, b; a + b; \cdot)$, which includes those appearing in (2.2), can be applied (see [2, 4]) to obtain inequalities directly relating these compound means.

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 12

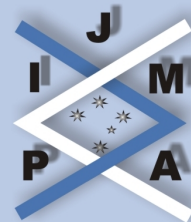
Go Back

Full Screen

Close

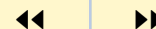
journal of **inequalities**
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mathematics

issn: 1443-5756



Title Page

Contents



Page 9 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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3. Applications

Borwein et al. (see [4, 5] and the references therein) used rather involved modular equations to discover means $\mathcal{M}_n, \mathcal{N}_n$ that can be used to build hypergeometric analogues $\mathcal{AG}_n \equiv \mathcal{M}_n \otimes \mathcal{N}_n$ converging quadratically to closed-form expressions involving ${}_2F_1(1/2 - s, 1/2 + s; 1; \cdot)$. In particular, they demonstrated that such compound means exist for $s = 0, 1/6, 1/4, 1/3$ (and the trivial case $s = 1/2$). The resulting closed forms include

$$\mathcal{AG}_2(1, r) = {}_2F_1(1/2, 1/2; 1; 1 - r^2)^{-1},$$

$$\mathcal{AG}_3(1, r) = {}_2F_1(1/3, 2/3; 1; 1 - r^3)^{-1},$$

$$\mathcal{AG}_4(1, r) = {}_2F_1(1/4, 3/4; 1; 1 - r^2)^{-2},$$

$$\mathcal{AG}_6(1, r) = {}_2F_1(1/6, 5/6; 1; 1 - r^3)^{-2}.$$

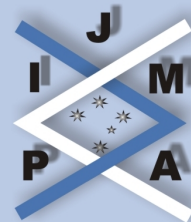
Notice that each ${}_2F_1$ satisfies the form appearing in Corollary 2.2. It can be shown that $\mathcal{AG}_2, \mathcal{AG}_3$, and \mathcal{AG}_4 are formed by compounding the following homogeneous means:

$$\mathcal{M}_2(a, b) \equiv \frac{a+b}{2}, \quad \mathcal{N}_2(a, b) \equiv \sqrt{ab},$$

$$\mathcal{M}_3(a, b) \equiv \frac{a+2b}{3}, \quad \mathcal{N}_3(a, b) \equiv \sqrt[3]{\frac{b(a^2 + ba + b^2)}{3}},$$

$$\mathcal{M}_4(a, b) \equiv \frac{a+3b}{4}, \quad \mathcal{N}_4(a, b) \equiv \sqrt{\frac{b(a+b)}{2}}.$$

(See [5] for the development of these and the more intricate $\mathcal{M}_6, \mathcal{N}_6$.) Applying Corollary 2.2 with $\alpha = 1/3, p = 3$, and invoking homogeneity with $r = b/a$, we



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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find

$$\mathcal{A}_\lambda \left(\frac{1}{3}; a, b \right) < \mathcal{AG}_3(a, b) < \mathcal{A}_\mu \left(\frac{1}{3}; a, b \right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 1$. In a similar fashion, with $\alpha = 1/4$ and $p = 2$, (2.2) implies

$$\mathcal{A}_\lambda \left(\frac{1}{4}; a, b \right) < \mathcal{AG}_4(a, b) < \mathcal{A}_\mu \left(\frac{1}{4}; a, b \right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 3/4$. Since $\mathcal{A}_{3/4}(1/4; a, b) < \mathcal{A}_1(1/4; a, b) = M_4(a, b)$, this sharpens the known fact that $\mathcal{AG}_4(a, b) < M_4(a, b)$. Next, with $\alpha = 1/6$ and $p = 3$, Corollary 2.2 yields

$$\mathcal{A}_\lambda \left(\frac{1}{6}; a, b \right) < \mathcal{AG}_6(a, b) < \mathcal{A}_\mu \left(\frac{1}{6}; a, b \right) \quad \text{for all } a > b > 0,$$

if and only if $\lambda \leq 0$ and $\mu \geq 5/4$. Finally, we note that another proof of the sharpness of (1.1) can be obtained by applying Corollary 2.2 with $\alpha = 1/2$ and $p = 2$.

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Inequalities for Hypergeometric

Analogues of the AGM

Roger W. Barnard and

Kendall C. Richards

vol. 8, iss. 3, art. 65, 2007

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 11 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756

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Inequalities for Hypergeometric

Analogues of the AGM

Roger W. Barnard and

Kendall C. Richards

vol. 8, iss. 3, art. 65, 2007

Title Page

Contents



Page 12 of 12

Go Back

Full Screen

Close

journal of **inequalities**
in pure and applied
mathematics

issn: 1443-5756