journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 8 (2007), Issue 2, Article 60, 5 pp.

ON CERTAIN INEQUALITIES IMPROVING THE HERMITE-HADAMARD INEQUALITY

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Received 21 April, 2007; accepted 04 June, 2007
Communicated by S.S. Dragomir

ABSTRACT. A generalized form of the Hermite-Hadamard inequality for convex Lebesgue integrable functions are obtained.

Key words and phrases: Convex function, Hermite-Hadamard inequality, Mean value.

2000 *Mathematics Subject Classification.* Primary 26A51; Secondary 26A46, 26A48.

The classical Hermite-Hadamard inequality gives us an estimate, from below and from above, of the mean value of a convex function $f : [a, b] \to \mathbb{R}$:

$$\text{(HH)} \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

See [2, pp. 50-51], for details. This result can be easily improved by applying (HH) on each of the subintervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$; summing up side by side we get

$$\text{(SLHH)} \qquad \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\text{(SRHH)} \qquad \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

Usually, the precision in the (HH) inequalities is estimated via Ostrowski's and Iyengar's inequalities. See [2], p. 63 and respectively p. 191, for details. Based on previous work done by S.S. Dragomir and A.McAndrew [1], we shall prove here several better results, that apply to a slightly larger class of functions.

We start by estimating the deviation of the support line of a convex function from the mean value. The main ingredient is the existence of the subdifferential.

The authors thank to Constantin P. Niculescu and Josip Pečarić for many valuable suggestions.
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Theorem 1. *Assume that f is Lebesgue integrable and convex on (a,b). Then*

$$\frac{1}{b-a}\int_a^b f(y)dy + \varphi(x)\left(x-\frac{a+b}{2}\right) - f(x) \geq \left|\frac{1}{b-a}\int_a^b |f(y)-f(x)|\,dy - |\varphi(x)|\,\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right|$$

for all $x \in (a,b)$.

Here $\varphi : (a,b) \to \mathbb{R}$ is any function such that $\varphi(x) \in [f'_-(x), f'_+(x)]$ for all $x \in (a,b)$.

Proof. In fact,

$$f(y) \geq f(x) + (y-x)\varphi(x)$$

for all $x, y \in (a,b)$, which yields

(Sd) $$f(y) - f(x) - (y-x)\varphi(x) = |f(y)-f(x)-(y-x)\varphi(x)| \geq ||f(y)-f(x)| - |y-x|\,|\varphi(x)||\,.$$

By integrating side by side we get

$$\begin{aligned}\int_a^b f(y)dy - (b-a)f(x) + (b-a)\left(x-\frac{a+b}{2}\right)\varphi(x) &\geq \int_a^b ||f(y)-f(x)| - |y-x|\,|\varphi(x)||\,dy \\ &\geq \left|\int_a^b |f(y)-f(x)|\,dy - |\varphi(x)|\int_a^b |y-x|\,dy\right| \\ &= \left|\int_a^b |f(y)-f(x)|\,dy - |\varphi(x)|\,\frac{(x-a)^2+(b-x)^2}{2}\right|\end{aligned}$$

and it remains to simplify both sides by $b-a$. □

Theorem 1 applies for example to convex functions not necessarily defined on compact intervals, for example, to $f(x) = (1-x^2)^{-\alpha}$, $x \in (-1,1)$, for $\alpha \geq 0$.

Theorem 2. *Assume that $f : [a,b] \to \mathbb{R}$ is a convex function. Then*

$$\frac{1}{2}\left[f(x) + \frac{f(b)(b-x)+f(a)(x-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(y)dy \geq \frac{1}{2}\left|\frac{1}{b-a}\int_a^b |f(x)-f(y)|\,dy - \frac{1}{b-a}\int_a^b |x-y|\,|f'(y)|\,dy\right|$$

for all $x \in (a,b)$.

Proof. Without loss of generality we may assume that f is also continuous. See [2, p. 22] (where it is proved that f admits finite limits at the endpoints).

In this case f is absolutely continuous and thus it can be recovered from its derivative. The function f is differentiable except for countably many points, and letting $\mathcal{E}$ denote this exceptional set, we have

$$f(x) \geq f(y) + (x-y)f'(y)$$

for all $x \in [a,b]$ and all $y \in [a,b]\backslash\mathcal{E}$.This yields

$$f(x) - f(y) - (x-y)f'(y) = |f(x)-f(y)-(x-y)f'(y)| \geq ||f(x)-f(y)| - |x-y|\cdot|f'(y)||\,,$$

so that by integrating side by side with respect to y we get

$$(b-a)f(x) - 2\int_a^b f(y)dy + f(b)(b-x) + f(a)(x-a) \\ \geq \left| \int_a^b |f(x)-f(y)|\,dy - \int_a^b |x-y|\,|f'(y)|\,dy \right|$$

equivalently,

$$f(x) + \frac{f(b)(b-x)+f(a)(x-a)}{b-a} - \frac{2}{b-a}\int_a^b f(y)dy \\ \geq \frac{1}{b-a}\left| \int_a^b |f(x)-f(y)|\,dy - \int_a^b |x-y|\,|f'(y)|\,dy \right|$$

and the result follows. □

A variant of Theorem 2, in the case where f is convex only on (a,b), is as follows:

Theorem 3. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotone on* $[a,b]$ *and convex on* (a,b). *Then*

$$\frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(y)dy \\ \geq \left| \frac{1}{b-a}\int_a^b \operatorname{sgn}(x-y)f(y)dy \right. \\ \left. + \frac{1}{2(b-a)}\left[f(x)(a+b-2x) + (x-a)f(a) + (b-x)f(b)\right] \right|$$

for all $x \in (a,b)$.

Proof. Consider for example the case where f is nondecreasing on $[a,b]$. Then

$$\begin{aligned} \int_a^b |f(x)-f(y)|\,dy &= \int_a^x |f(x)-f(y)|\,dy + \int_x^b |f(x)-f(y)|\,dy \\ &= (x-a)f(x) - \int_a^x f(y)dy + \int_x^b f(y)dy - (b-x)f(x) \\ &= (2x-a-b)f(x) - \int_a^x f(y)dy + \int_x^b f(y)dy. \end{aligned}$$

As in the proof of Theorem 2, we may restrict ourselves to the case where f is absolutely continuous, which yields

$$\begin{aligned} \int_a^b |x-y|\,|f'(y)|\,dy &= \int_a^x (x-y)f'(y)dy + \int_x^b (y-x)f'(y)dy \\ &= (a-x)f(a) + (b-x)f(b) + \int_a^x f(y)dy - \int_x^b f(y)dy. \end{aligned}$$

By Theorem 2, we conclude that

$$\frac{1}{2}\left[f(y) + \frac{f(b)(b-y)+f(a)(y-a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(x)dx$$

$$\geq \frac{1}{2}\left|\frac{2}{b-a}\left[\int_x^b f(y)dy - \int_a^x f(y)dy\right] + \frac{f(x)(2x-a-b)}{b-a} - \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right|.$$

The case where f is nonincreasing can be treated in a similar way. ☐

For $x = (a+b)/2$, Theorem 3 gives us

$$\text{(UE)}\quad \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] - \frac{1}{b-a}\int_a^b f(y)dy \geq \left|\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-y\right) f(y)dy + \frac{f(a)+f(b)}{4}\right|,$$

which in the case of the exponential function means

$$\frac{1}{2}\left[\exp\frac{a+b}{2} + \frac{\exp a + \exp b}{2}\right] - \frac{\exp b - \exp a}{b-a} \geq \left|\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(\frac{a+b}{2}-y\right)\exp y\, dy + \frac{\exp a + \exp b}{4}\right|$$

for all $a, b \in \mathbb{R}$, $a < b$, equivalently,

$$\frac{1}{2}\left[\sqrt{ab} + \frac{a+b}{2}\right] - \frac{b-a}{\ln b - \ln a} \geq \left|\frac{a+b}{4} - \frac{a+b-2\sqrt{ab}}{\ln b - \ln a}\right|$$

for all $0 < a < b$.

This represents an improvement on *Polya's inequality*,

$$\text{(Po)}\qquad \frac{2}{3}\cdot\sqrt{ab} + \frac{1}{3}\cdot\frac{a+b}{2} > \frac{b-a}{\ln b - \ln a}$$

since

$$\frac{2}{3}\cdot\sqrt{ab} + \frac{1}{3}\cdot\frac{a+b}{2} > \frac{1}{2}\sqrt{ab} + \frac{a+b-2\sqrt{ab}}{\ln b - \ln a}.$$

In fact, the last inequality can be restated as

$$(x+1)^2 \ln x > 3\,(x-1)^2$$

for all $x > 1$, a fact that can be easily checked using calculus.

As Professor Niculescu has informed us, we can embed Polya's inequality into a long sequence of interpolating inequalities involving the geometric, the arithmetic, the logarithmic and the identric means:

$$\begin{aligned}\sqrt{ab} &< \left(\sqrt{ab}\right)^{2/3}\left(\frac{a+b}{2}\right)^{1/3} \\ &< \frac{b-a}{\ln b - \ln a} < \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \\ &< \frac{2}{3}\cdot\sqrt{ab} + \frac{1}{3}\cdot\frac{a+b}{2} \\ &< \sqrt{\frac{a+b}{2}\sqrt{ab}} \\ &< \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right) < \frac{a+b}{2}\end{aligned}$$

for all $0 < a < b$.

Remark 4. The extension of Theorems 1 – 3 above to the context of weighted measures is straightforward and we shall omit the details. However, the problem of estimating the Hermite-Hadamard inequality in the case of several variables is left open.

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