

NEW GENERAL INTEGRAL OPERATORS OF p -VALENT FUNCTIONS

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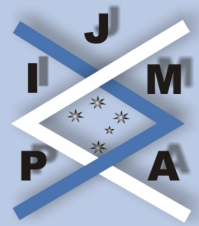
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Key words: Analytic functions, p -valent starlike, convex and close-to-convex functions, Uniformly p -valent close-to-convex functions, Strongly starlike, Integral operator.

Abstract: In this paper, we introduce new general integral operators. New sufficient conditions for these operators to be p -valently starlike, p -valently close-to-convex, uniformly p -valent close-to-convex and strongly starlike of order γ ($0 < \gamma \leq 1$) in the open unit disk are obtained.



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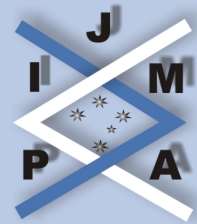
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1. Introduction and Definitions

Let \mathcal{A}_p denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} \in \{1, 2, \dots\}),$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. We write $\mathcal{A}_1 = \mathcal{A}$. A function $f \in \mathcal{A}_p$ is said to be p -valently starlike of order β ($0 \leq \beta < p$) if and only if

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \beta \quad (z \in \mathcal{U}).$$

We denote by $\mathcal{S}_p^*(\beta)$, the class of all such functions. On the other hand, a function $f \in \mathcal{A}_p$ is said to be p -valently convex of order β ($0 \leq \beta < p$) if and only if

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > \beta \quad (z \in \mathcal{U}).$$

Let $\mathcal{K}_p(\beta)$ denote the class of all those functions which are p -valently convex of order β in \mathcal{U} . Furthermore, a function $f(z) \in \mathcal{A}_p$ is said to be in the subclass $\mathcal{C}_p(\beta)$ of p -valently close-to-convex functions of order β ($0 \leq \beta < p$) in \mathcal{U} if and only if

$$\operatorname{Re} \left(\frac{f'(z)}{z^{p-1}} \right) > \beta \quad (z \in \mathcal{U}).$$

Note that $\mathcal{S}_p^*(0) = \mathcal{S}_p^*$, $\mathcal{K}_p(0) = \mathcal{K}_p$ and $\mathcal{C}_p(0) = \mathcal{C}_p$ are, respectively, the classes of p -valently starlike, p -valently convex and p -valently close-to-convex functions in \mathcal{U} . Also, we note that $\mathcal{S}_1^* = \mathcal{S}^*$, $\mathcal{K}_1 = \mathcal{K}$ and $\mathcal{C}_1 = \mathcal{C}$ are, respectively, the usual classes of starlike, convex and close-to-convex functions in \mathcal{U} .



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A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{UC}_p(\beta)$ of uniformly p -valent close-to-convex functions of order β ($0 \leq \beta < p$) in \mathcal{U} if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{g(z)} - \beta \right) \geq \left| \frac{zf'(z)}{g(z)} - p \right| \quad (z \in \mathcal{U}),$$

for some $g(z) \in \mathcal{US}_p(\beta)$, where $\mathcal{US}_p(\beta)$ is the class of uniformly p -valent starlike functions of order β ($-1 \leq \beta < p$) in \mathcal{U} and satisfies

$$(1.1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} - \beta \right) \geq \left| \frac{zf'(z)}{f(z)} - p \right| \quad (z \in \mathcal{U}).$$

Uniformly p -valent starlike functions were first introduced in [10].

For $\alpha_i > 0$ and $f_i \in \mathcal{A}_p$, we define the following general integral operators

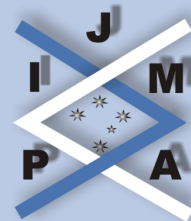
$$(1.2) \quad F_p(z) = \int_0^z pt^{p-1} \left(\frac{f_1(t)}{t^p} \right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p} \right)^{\alpha_n} dt$$

and

$$(1.3) \quad G_p(z) = \int_0^z pt^{p-1} \left(\frac{f'_1(t)}{pt^{p-1}} \right)^{\alpha_1} \cdots \left(\frac{f'_n(t)}{pt^{p-1}} \right)^{\alpha_n} dt.$$

If we take $p = 1$, we obtain of the general integral operators $F_1(z) = F_n(z)$ and $G_1(z) = F_{\alpha_1, \dots, \alpha_n}(z)$ introduced and studied by Breaz and Breaz [3] and Breaz et al. [6] (see also [2, 4, 8, 9]). Also for $p = n = 1$, $\alpha_1 = \alpha \in [0, 1]$ in (1.2), we obtain the integral operator $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ studied in [12] and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| \leq 1/4$ in (1.3), we obtain the integral operator $\int_0^z (f'(t))^\alpha dt$ studied in [11, 15].

There are many papers in which various sufficient conditions for multivalent starlikeness have been obtained. In this paper, we derive new sufficient conditions for the



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operators $F_p(z)$ and $G_p(z)$ to be p -valently starlike, p -valently close-to-convex and uniformly p -valent close-to-convex in \mathcal{U} . Furthermore, we give new sufficient conditions for these two general operators to be strongly starlike of order γ ($0 < \gamma \leq 1$) in \mathcal{U} .

In order to derive our main results, we have to recall here the following results:

Lemma 1.1 ([13]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < p + \frac{1}{4} \quad (z \in \mathcal{U}),$$

then f is p -valently starlike in \mathcal{U} .

Lemma 1.2 ([7]). *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} + 1 - p \right| < p + 1 \quad (z \in \mathcal{U}),$$

then f is p -valently starlike in \mathcal{U} .

Lemma 1.3 ([16]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < p + \frac{a+b}{(1+a)(1-b)} \quad (z \in \mathcal{U}),$$

where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then f is p -valently close-to-convex in \mathcal{U} .

Lemma 1.4 ([1]). *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < p + \frac{1}{3} \quad (z \in \mathcal{U}),$$

then f is uniformly p -valent close-to-convex in \mathcal{U} .



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Lemma 1.5 ([17]). If $f \in \mathcal{A}_p$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{p}{4} - 1 \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \sqrt{\frac{zf'(z)}{f(z)}} > \frac{\sqrt{p}}{2} \quad (z \in \mathcal{U}).$$

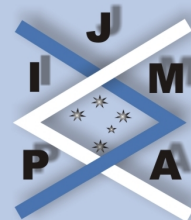
Lemma 1.6 ([14]). If $f \in \mathcal{A}_p$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > p - \frac{\gamma}{2} \quad (z \in \mathcal{U}),$$

then

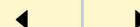
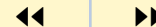
$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\pi}{2} \gamma \quad (0 < \gamma \leq 1; z \in \mathcal{U}),$$

or f is strongly starlike of order γ in \mathcal{U} .



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2. Sufficient Conditions for the Operator F_p

We begin by establishing sufficient conditions for the operator F_p to be in \mathcal{S}_p^* .

Theorem 2.1. Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies

$$(2.1) \quad \operatorname{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right) < p + \frac{1}{4 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then F_p is p -valently starlike in \mathcal{U} .

Proof. From the definition (1.2), we observe that $F_p(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$(2.2) \quad F_p'(z) = p z^{p-1} \left(\frac{f_1(z)}{z^p} \right)^{\alpha_1} \dots \left(\frac{f_n(z)}{z^p} \right)^{\alpha_n}.$$

Differentiating (2.2) logarithmically and multiplying by z , we obtain

$$\frac{z F_p''(z)}{F_p'(z)} = (p-1) + \sum_{i=1}^n \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - p \right).$$

Thus we have

$$(2.3) \quad 1 + \frac{z F_p''(z)}{F_p'(z)} = p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} \right).$$

Taking the real part of both sides of (2.3), we have

$$(2.4) \quad \operatorname{Re} \left(1 + \frac{z F_p''(z)}{F_p'(z)} \right) = p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \operatorname{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right).$$



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From (2.4) and (2.1), we obtain

$$(2.5) \quad \operatorname{Re} \left(1 + \frac{zF_p''(z)}{F_p'(z)} \right) < p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(p + \frac{1}{4 \sum_{i=1}^n \alpha_i} \right) \\ = p + \frac{1}{4}.$$

Hence by Lemma 1.1, we get $F_p \in \mathcal{S}_p^*$. This completes the proof. \square

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.1, we have:

Corollary 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < 1 + \frac{1}{4\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ is starlike in \mathcal{U} .

In the next theorem, we derive another sufficient condition for p -valently starlike functions in \mathcal{U} .

Theorem 2.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$(2.6) \quad \left| \frac{zf_i'(z)}{f_i(z)} - p \right| < \frac{p+1}{\sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then F_p is p -valently starlike in \mathcal{U} .

Proof. From (2.3) and the hypotheses (2.6), we have

$$\begin{aligned} \left| 1 + \frac{zF_p''(z)}{F_p'(z)} - p \right| &= \left| \sum_{i=1}^n \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - p \right) \right| \\ &< \sum_{i=1}^n \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - p \right| \\ &< \sum_{i=1}^n \alpha_i \left(\frac{p+1}{\sum_{i=1}^n \alpha_i} \right) = p+1. \end{aligned}$$

Now using Lemma 1.2, we immediately get $F_p \in \mathcal{S}_p^*$. □

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.3, we have:

Corollary 2.4. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{2}{\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ is starlike in \mathcal{U} .

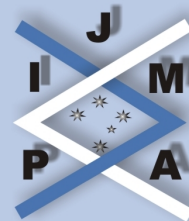
Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for F_p to be p -valently close-to-convex and uniformly p -valent close-to-convex in \mathcal{U} .

Theorem 2.5. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$(2.7) \quad \operatorname{Re} \left(\frac{zf_i'(z)}{f_i(z)} \right) < p + \frac{(a+b)}{(1+a)(1-b) \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then F_p is p -valently close-to-convex in \mathcal{U} .

Proof. From (2.4) and the hypotheses (2.7) and applying Lemma 1.3, we have $F_p \in \mathcal{C}_p(\beta)$. □



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Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.5, we have:

Corollary 2.6. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < 1 + \frac{(a+b)}{(1+a)(1-b)\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ is close-to-convex in \mathcal{U} .

Theorem 2.7. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$(2.8) \quad \operatorname{Re} \left(\frac{zf'_i(z)}{f_i(z)} \right) < p + \frac{1}{3 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then F_p is uniformly p -valent close-to-convex in \mathcal{U} .

Proof. The proof of the theorem follows by applying Lemma 1.4 and using (2.4), (2.8) to get $F_p \in \mathcal{UC}_p(\beta)$. \square

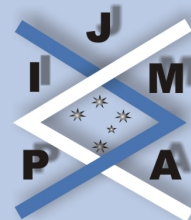
Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 2.7, we have:

Corollary 2.8. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) < 1 + \frac{1}{3\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ is uniformly close-to-convex in \mathcal{U} .

Using Lemma 1.5, we obtain the next result



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Theorem 2.9. Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies

$$(2.9) \quad \operatorname{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right) > p - \frac{3p+4}{4 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \sqrt{\frac{z F_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \quad (z \in \mathcal{U}).$$

Proof. It follows from (2.4) and (2.9) that

$$\operatorname{Re} \left(1 + \frac{z F_p''(z)}{F_p'(z)} \right) > p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(p - \frac{3p+4}{4 \sum_{i=1}^n \alpha_i} \right) = \frac{p}{4} - 1.$$

By Lemma 1.5, we conclude that

$$\operatorname{Re} \sqrt{\frac{z F_p'(z)}{F_p(z)}} > \frac{\sqrt{p}}{2} \quad (z \in \mathcal{U}).$$

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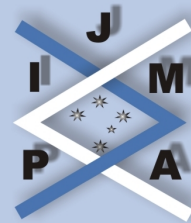
Letting $n = p = 1$, $\alpha_1 = 1$ and $f_1 = f$ in Theorem 2.9, we have:

Corollary 2.10. If $f \in \mathcal{A}$ satisfies

$$(2.10) \quad \operatorname{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right) > -\frac{3}{4} \quad (z \in \mathcal{U}),$$

then

$$(2.11) \quad \operatorname{Re} \sqrt{\frac{f(z)}{\int_0^z \left(\frac{f(t)}{t} \right) dt}} > \frac{1}{2} \quad (z \in \mathcal{U}).$$



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3. Sufficient Conditions for the Operator G_p

The first two theorems in this section give a sufficient condition for the integral operator G_p to be in the class \mathcal{S}_p^* .

Theorem 3.1. Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies

$$(3.1) \quad \operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) < p + \frac{1}{4 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then G_p is p -valently starlike in \mathcal{U} .

Proof. From the definition (1.3), we observe that $G_p(z) \in \mathcal{A}_p$ and

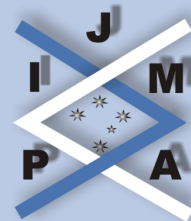
$$\frac{z G_p''(z)}{G_p'(z)} = (p-1) + \sum_{i=1}^n \alpha_i \left(\frac{z f_i''(z)}{f_i'(z)} - (p-1) \right)$$

or

$$(3.2) \quad 1 + \frac{z G_p''(z)}{G_p'(z)} = p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right).$$

Taking the real part of both sides of (3.2), we have

$$(3.3) \quad \operatorname{Re} \left(1 + \frac{z G_p''(z)}{G_p'(z)} \right) = p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right).$$



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From (3.3) and the hypotheses (3.1), we obtain

$$(3.4) \quad \operatorname{Re} \left(1 + \frac{zG_p'''(z)}{G_p'(z)} \right) < p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(p + \frac{1}{4 \sum_{i=1}^n \alpha_i} \right) \\ = p + \frac{1}{4}.$$

Therefore, using Lemma 1.1, it follows that the integral operator G_p belongs to the class \mathcal{S}_p^* . \square

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.1, we obtain

Corollary 3.2. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{1}{4\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is starlike in \mathcal{U} .

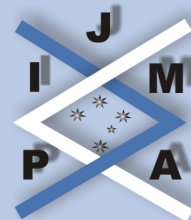
Theorem 3.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$(3.5) \quad \left| \frac{zf_i''(z)}{f_i'(z)} \right| < \frac{p+1}{\sum_{i=1}^n \alpha_i} - p + 1 \quad (z \in \mathcal{U}),$$

where $\sum_{i=1}^n \alpha_i > 1$, then G_p is p -valently starlike in \mathcal{U} .

Proof. It follows from (3.2) and (3.5) that

$$\left| 1 + \frac{zG_p'''(z)}{G_p'(z)} - p \right| = \left| \sum_{i=1}^n \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)} \right) - (p-1) \sum_{i=1}^n \alpha_i \right| \\ < (p-1) \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \alpha_i \left(\frac{p+1}{\sum_{i=1}^n \alpha_i} - p + 1 \right) < p + 1.$$



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Therefore, it follows from Lemma 1.2 that G_p is in the class S_p^* . □

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.3, we obtain:

Corollary 3.4. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \right| < \frac{2}{\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is starlike in \mathcal{U} .

Applying Lemmas 1.3 and 1.4, we obtain the following sufficient conditions for G_p to be p -valently close-to-convex and uniformly p -valent close-to-convex in \mathcal{U} .

Theorem 3.5. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$(3.6) \quad \operatorname{Re} \left(1 + \frac{zf_i''(z)}{f_i'(z)} \right) < p + \frac{a+b}{(1+a)(1-b) \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

where $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then G_p is p -valently close-to-convex in \mathcal{U} .

Proof. In view of (3.3) and (3.6) and by using Lemma 1.3, we have $G_p \in \mathcal{C}_p(\beta)$. □

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 3.5, we obtain

Corollary 3.6. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{a+b}{(1+a)(1-b)\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, $a > 0$, $b \geq 0$ and $a + 2b \leq 1$, then $\int_0^z (f'(t))^\alpha dt$ is close-to-convex in \mathcal{U} .



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Theorem 3.7. Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies

$$(3.7) \quad \operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) < p + \frac{1}{3 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then G_p is uniformly p -valent close-to-convex in \mathcal{U} .

Proof. In view of (3.3) and (3.7) and by using Lemma 1.4, we have $G_p \in \mathcal{UC}_p(\beta)$. \square

Letting $n = p = \alpha = 1$ and $f_1 = f$ in Theorem 3.7, we have:

Corollary 3.8. If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) < 1 + \frac{1}{3\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is uniformly close-to-convex in \mathcal{U} .

Using Lemma 1.5, we obtain the next result.

Theorem 3.9. Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies

$$(3.8) \quad \operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) > p - \frac{3p+4}{4 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then

$$(3.9) \quad \operatorname{Re} \sqrt{\frac{z G_p'(z)}{G_p(z)}} > \frac{\sqrt{p}}{2} \quad (z \in \mathcal{U}).$$



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Proof. It follows from (3.3) and (3.8) that

$$\operatorname{Re} \left(1 + \frac{zG_p'''(z)}{G_p'(z)} \right) > p \left(1 - \sum_{i=1}^n \alpha_i \right) + \sum_{i=1}^n \alpha_i \left(p - \frac{3p+4}{4 \sum_{i=1}^n \alpha_i} \right) = \frac{p}{4} - 1.$$

By Lemma 1.5, we get the result (3.9). □

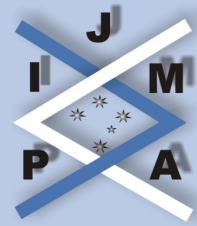
Letting $n = p = 1$, $\alpha_1 = 1$ and $f_1 = f$ in Theorem 3.9, we have

Corollary 3.10. *If $f \in \mathcal{A}$ satisfies*

$$(3.10) \quad \operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) > -\frac{3}{4} \quad (z \in \mathcal{U}),$$

then

$$(3.11) \quad \operatorname{Re} \sqrt{\frac{z f'(z)}{\int_0^z f'(t) dt}} > \frac{1}{2} \quad (z \in \mathcal{U}).$$



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4. Strong Starlikeness of the Operators F_p and G_p

Applying Lemma 1.6 and using (2.4), we obtain the following sufficient condition for the operator F_p to be strongly starlike of order γ in \mathcal{U} .

Theorem 4.1. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\operatorname{Re} \left(\frac{z f_i'(z)}{f_i(z)} \right) > p - \frac{\gamma}{2 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then F_p is strongly starlike of order γ ($0 < \gamma \leq 1$) in \mathcal{U} .

Letting $n = p = 1$, $\alpha_1 = \alpha$ and $f_1 = f$ in Theorem 4.1, we have

Corollary 4.2. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > 1 - \frac{\gamma}{2\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt$ is strongly starlike of order γ ($0 < \gamma \leq 1$) in \mathcal{U} .

Applying once again Lemma 1.6 and using (3.3), we obtain the following sufficient condition for the operator G_p to be strongly starlike of order γ in \mathcal{U} .

Theorem 4.3. *Let $\alpha_i > 0$ be real numbers for all $i = 1, 2, \dots, n$. If $f_i \in \mathcal{A}_p$ for all $i = 1, 2, \dots, n$ satisfies*

$$\operatorname{Re} \left(1 + \frac{z f_i''(z)}{f_i'(z)} \right) > p - \frac{\gamma}{2 \sum_{i=1}^n \alpha_i} \quad (z \in \mathcal{U}),$$

then G_p is strongly starlike of order γ ($0 < \gamma \leq 1$) in \mathcal{U} .



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Letting $n = p = \alpha_1 = 1$ and $f_1 = f$ in Theorem 4.3, we have

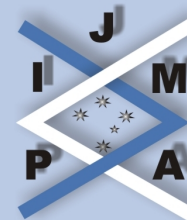
Corollary 4.4. *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 1 - \frac{\gamma}{2\alpha} \quad (z \in \mathcal{U}),$$

where $\alpha > 0$, then $\int_0^z (f'(t))^\alpha dt$ is strongly starlike of order γ ($0 < \gamma \leq 1$) in \mathcal{U} .

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