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CARLEMAN'S INEQUALITY - HISTORY, PROOFS AND SOME NEW GENERALIZATIONS

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ABSTRACT. Carleman's inequality reads

$$a_1+\sqrt{a_1a_2}+...+\sqrt[k]{a_1...a_k}<e\left(a_1+a_2+....\right),$$

where a_k , $k=1,2,....$, are positive numbers. In this paper we present some simple proofs of and several remarks (e.g. historical) about the inequality and its corresponding continuous version. Moreover, we discuss and comment on some very new results. We also include some new proofs and results e.g. a weight characterization of a general weighted Carleman type inequality for the case $0<\mathrm{p}\leq\mathrm{q}<\infty$. We also include some facts about T. Carleman and his work.

Key words and phrases: Inequalities, Carleman's inequality, Pólya-Knopp's inequality, Sharp constants, Proofs, Weights, Historical remarks.

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1. INTRODUCTION

In this paper we discuss the following remarkable inequality:

$$a_1+\sqrt{a_1a_2}+\cdots+\sqrt[k]{a_1a_2\cdots a_k}<e\left(a_1+a_2+\cdots\right), \tag{1.1}$$

where $a_1,\ a_2,$ ... are positive numbers and $\sum_{i=1}^{\infty}a_i$ is convergent. This inequality was presented in 1922 in [8] by the Swedish mathematician Torsten Carleman (1892-1942) and it is called Carleman's inequality. Carleman discovered this inequality during his important work

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on quasianalytical functions and he could hardly have imagined at that time that this discovery would be an object of such great interest. The continuous version of (1.1) reads

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx < e\int_0^\infty f(x)dx, \tag{1.2}$$

where $f(t) > 0$ and it is sometimes called Knopp's inequality with reference to [32] (cf. Remark 3.2). However, it seems that it was G. Pólya who first discovered this inequality (see Remark 2.3). Therefore we prefer to call it *Pólya-Knopp's inequality*.

In Section 2 of this paper we present several proofs of and remarks on (1.1). In Section 3 we prove that (1.2) implies (1.1) and present some proofs of (1.2) (and thus some more proofs of (1.1)).

In Section 4 we give some examples of recently published generalizations of (1.1) and (1.2). We discuss and comment on these results and put them into the frame presented above. We also include some new proofs and results, namely, we prove a new weight characterization of a general weighted Carleman type inequality for the case $0 < p \leq q < \infty$, i.e., we prove a necessary and sufficient condition on the sequences $\{b_k\}_{k=1}^\infty$ and $\{d_k\}_{k=1}^\infty$ so that the inequality

$$\left(\sum_{k=1}^\infty \left(\sqrt[k]{a_1 a_2 \cdots a_k}\right)^q b_k\right)^{\frac{1}{q}} \leq C\left(\sum_{k=1}^\infty a_k^p d_k\right)^{\frac{1}{p}} \tag{1.3}$$

holds for some finite and positive constant C and for all sequences $\{a_k\}_{k=1}^\infty$of non-negative numbers. Moreover, we give upper and lower estimates of the best constant C in the inequality (the corresponding operator norm). Finally, we include some facts about Torsten Carleman and his work, which we have found, for example, by studying [31] and [58] and this partly complements the information given by Professor Lars Gårding in his excellent description in [19].

2. Some Proofs of (1.1)

Proof 1. (Rough sketch of Carleman's original proof) Carleman first noted that the problem can be solved by finding a maximum of the expression

$$\sum_{i=1}^k (a_1 a_2 \cdots a_i)^{\frac{1}{i}}$$

under the constraint

$$\sum_{i=1}^k a_i = 1.$$

He then substituted $a_i = e^{-x_i}$ and obtained the simpler problem:
Find a maximum M_k for $k = 1, 2, \dots$ of

$$G = \sum_{i=1}^k e^{-\frac{x_1+x_2+\cdots+x_i}{i}}$$

under the constraint

$$H = \sum_{i=1}^k e^{-x_i} = 1.$$

This problem can be solved by using the Lagrange multiplier method. Unfortunately this leads to some technical calculations, which of course Carleman carried out in an elegant way. We leave out these calculations here, and only refer to Carleman's paper [8], where all the details are presented. The result is that $M_k < e$ for all $k \in \mathbb{Z}_+$. Carleman then showed separately that the inequality (1.1) is strict when the sum on the left hand side converges.

□

Remark 2.1. In the same paper [8], Carleman proved that the inequality (1.1) does not hold in general for any constant $C < e$, i.e., that the constant e is sharp.

Proof 2. (via Hardy's inequality)
The discrete version of Hardy's inequality reads (see [21], [23])

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{i=1}^{k}a_i\right)^p < \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{\infty}a_k^p, \quad p>1. \tag{2.1}$$

Replace a_i with $a_i^{\frac{1}{p}}$ and note that by using that $x = e^{\ln x}$ and the definition of the derivative we find that

$$\begin{aligned}\left(\frac{1}{k}\sum_{i=1}^{k}a_i^{\frac{1}{p}}\right)^p &= \exp\frac{1}{p}\left(\ln\sum_{i=1}^{k}a_i^{\frac{1}{p}} - \ln\sum_{i=1}^{k}a_i^{0}\right)\\ &\to \exp\left(\left[D(\ln\sum_{i=1}^{k}a_i^{x})\right]_{x=0}\right) \qquad \text{(when } p\to\infty\text{)}\\ &= \exp\left(\left[\sum_{i=1}^{k}a_i^{x}\ln a_i\Big/\sum_{i=1}^{k}a_i^{x}\right]_{x=0}\right)\\ &= \exp\frac{1}{k}\sum_{i=1}^{k}\ln a_i = \left(\prod_{i=1}^{k}a_i\right)^{\frac{1}{k}}\end{aligned}$$

and we see that (2.1) leads to the non-strict inequality (1.1) since $\left(\frac{p}{p-1}\right)^p \to e$ when $p\to\infty$. Observe that this method does not automatically prove that we have strict inequality in (1.2) and this has to be proved separately (see for example our later proofs).

□

Remark 2.2. G.H. Hardy formulated his inequality (2.1) in 1920 in [20] and proved it in 1925 [21] but it seems that Carleman did not know about the inequality (2.1) at this time, since he does not refer to the simple connection that holds according to the proof above. This is somewhat remarkable since Carleman worked together with Hardy at that time, see for example their joint paper [9].

Remark 2.3. The above means that (1.1) may be considered as a limit inequality for the scale (2.1) of Hardy inequalities. This was pointed out by G.H. Hardy in 1925 in the paper [21, p. 156], but he pronounced that it was G. Pólya who made him aware of this interesting fact.

We now present two proofs which are based on variations of the arithmetic-geometric mean inequality (the AG-inequality).

Proof 3. We use the AG-inequality together with the fact that

$$\frac{(k+1)^k}{k!} = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)^2\cdots\left(1+\frac{1}{k}\right)^k < e^k \tag{2.2}$$

to obtain

$$\begin{aligned}\sum_{i=1}^{\infty} a_i &= \sum_{i=1}^{\infty} i a_i \sum_{k=i}^{\infty} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \sum_{i=1}^{k} i a_i \\ &= \sum_{k=1}^{\infty} \frac{a_1 + 2a_2 + \cdots + k a_k}{k(k+1)} \\ &> \sum_{k=1}^{\infty} \frac{1}{k+1} \left(k! \prod_{1}^{k} a_i \right)^{\frac{1}{k}} \\ &= \sum_{k=1}^{\infty} \left(\frac{k!}{(k+1)^k} \right)^{\frac{1}{k}} \left(\prod_{i=1}^{k} a_i \right)^{\frac{1}{k}} \geq \frac{1}{e} \sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} a_i \right)^{\frac{1}{k}}.\end{aligned}$$

This strict inequality holds, since we cannot have equality at the same time in all terms of the inequality. This can only occur if $a_k = \frac{c}{k}$ for some $c > 0$ but this cannot hold since $\sum_1^{\infty} a_k$ is convergent.

□

Remark 2.4. In the paper [20, p. 77], G.H. Hardy presented essentially this proof but he also pronounced that it was G. Knopp who pointed out this proof to him.

Proof 4. Because of the AG-inequality the following holds for every $i = 1, 2, \ldots$, every k and all $c_i > 0$:

$$\left(\prod_{1}^{k} a_i \right)^{\frac{1}{k}} = \left(\prod_{1}^{k} c_i \right)^{-\frac{1}{k}} \left(\prod_{1}^{k} c_i a_i \right)^{\frac{1}{k}} \leq \left(\prod_{1}^{k} c_i \right)^{-\frac{1}{k}} \frac{1}{k} \sum_{i=1}^{k} c_i a_i. \tag{2.3}$$

We now choose $c_i = \frac{(1+i)^i}{i^{i-1}}$, $i = 1, 2, \ldots, k$. Then

$$\left(\prod_{1}^{k} c_i \right)^{\frac{1}{k}} = k + 1 \tag{2.4}$$

and (2.3) and (2.4) give that

$$\begin{aligned}\sum_{k=1}^{\infty} \sqrt[k]{a_1 a_2 \cdots a_k} &\leq \sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} \sum_{i=1}^{k} c_i a_i \\ &= \sum_{i=1}^{\infty} c_i a_i \sum_{k=i}^{\infty} \frac{1}{k\,(k+1)} \\ &= \sum_{i=1}^{\infty} c_i a_i / i = \sum_{i=1}^{\infty} a_i \left(1 + \frac{1}{i} \right)^i \leq e \sum_{i=1}^{\infty} a_i.\end{aligned}$$

The strict inequality can be proved in a similar manner to Proof 3.

□

Remark 2.5. This proof was presented by G. Pólya (see [47, p. 249]) but here we follow the presentation which can be found in Professor Lars Hörmander's book [26, p. 24].

Proof 5. (Carleson's proof)
We first note that we can assume that $a_1 \geq a_2 \geq \cdots$ (because the sum on the left hand side of (1.1) obviously becomes the greatest if the sequence $\{a_i\}$ is rearranged in non-increasing order while the sum at the right hand side will be the same for every rearrangement). Let $m(x)$ be a polygon through the points $(0,0)$ and $\left(k, \sum_1^k \log(1/a_i)\right)$, $k = 1, 2, \dots$ The function $m(x)$ is obviously convex and because of that it holds that for every $r > 1$

$$m(rx) \geq m(x) + (r-1)xm'(x). \tag{2.5}$$

Furthermore

$$m'(x) = \log(1/a_k), \quad x \in (k-1, k), \tag{2.6}$$

and since $m(0) = 0$ and m is convex we have

$$\begin{aligned} \frac{m(x)}{x} &= \frac{m(x) - m(0)}{x} \leq \frac{m(k) - m(0)}{k} \\ &= \frac{m(k)}{k} = \frac{1}{k} \sum_1^k \log(1/a_i) \text{ for all } x \leq k. \end{aligned} \tag{2.7}$$

We now make a substitution and use Hölder's inequality and (2.5). Then, we find, for every $A > 0$ and $r > 1$,

$$\begin{aligned} \frac{1}{r} \int_0^A e^{-m(x)/x} dx &\leq \frac{1}{r} \int_0^{rA} e^{-m(x)/x} dx \\ &= \int_0^A e^{-m(rx)/rx} dx \\ &\leq \int_0^A e^{-m(x)/rx - ((r-1)/r)m'(x)} dx \\ &\leq \left(\int_0^A e^{-m(x)/x} dx \right)^{\frac{1}{r}} \left(\int_0^A e^{-m'(x)} dx \right)^{\frac{(r-1)}{r}} \end{aligned}$$

so that

$$\int_0^A e^{-m(x)/x} dx \leq r^{\frac{r}{r-1}} \int_0^A e^{-m'(x)} dx.$$

We let $A \to \infty, r \to 1+$ and note that $r^{\frac{r}{r-1}} \to e$ so that

$$\int_0^\infty e^{-m(x)/x} dx \leq e \int_0^\infty e^{-m'(x)} dx. \tag{2.8}$$

We now use (2.6) and (2.7) and get

$$\int_0^\infty e^{-m'(x)} dx = \sum_{k=1}^\infty \int_{k-1}^k e^{-m'(x)} dx = \sum_{k=1}^\infty e^{-\log(1/a_k)} = \sum_{k=1}^\infty a_k,$$

respectively,

$$\begin{aligned} \int_0^\infty e^{-m(x)/x} dx &= \sum_{k=1}^\infty \int_{k-1}^k e^{-m(x)/x} dx \\ &\geq \sum_{k=1}^\infty \exp\left(-\frac{1}{k} \sum_{i=1}^k \log\left(\frac{1}{a_i}\right)\right) = \sum_{k=1}^\infty \left(\prod_{i=1}^k a_i \right)^{\frac{1}{k}}. \end{aligned}$$

The non-strict inequality (1.1) follows by using these estimates and (2.8). The strict inequality follows from the fact that we cannot have equality in (2.7) at the same time for all x and k.

□

Remark 2.6. This is L. Carleson's proof (see [10]) and in fact he proved that the even more general inequality (2.8) holds for every piecewise differentiable convex function $m(x)$ on $[0,\infty]$ such that $m(0)=0$. In fact, Carleson formulated his inequality in the following slightly more general form:

$$\int_0^\infty x^k e^{-m(x)/x}dx \le e^{k+1}\int_0^\infty x^k e^{-m'(x)}dx, k>-1. \tag{2.9}$$

Proof 6. (via Redheffer's inequality) R.M. Redheffer proved in 1967 the following interesting inequality (see [48] and also [49]):

$$nG_n + \sum_{k=1}^n k\left(b_k-1\right)G_k \le \sum_{k=1}^n a_k b_k^k, \tag{2.10}$$

which holds for all $n=1,2,\dots$ and all positive sequences $\{b_k\}$ and where $G_k=\left(\prod_{i=1}^k a_i\right)^{\frac{1}{k}}$. In particular, we see that if

a) $b_k=1, k=1,2,\dots$, then $G_n \le \frac{1}{n}\sum_{k=1}^n a_k = A_n$, i.e. the AG-inequality,

b) $b_k = 1+\frac{1}{k}, k=1,2,\dots$, then $nG_n+\sum_{k=1}^n G_k \le \sum_{k=1}^n \left(1+\frac{1}{k}\right)^k a_k$,

which implies that the non-strict inequality (1.1) follows when $n\to\infty$. The strict inequality can also be proved by using the arguments in the following proof of the inequality (2.10). We use the elementary inequality

$$1+a\left(x-1\right)\le x^a,\ x>0, a>1, \tag{2.11}$$

(a simple proof of (2.11) can be obtained by putting $\alpha=\frac{1}{a}$ and replacing x with x^a in the following form of the AG-inequality: $x^\alpha 1^{1-\alpha}\le \alpha x+(1-\alpha)1$). We now use (2.11) with $a=k$ and $x=\frac{G_k}{G_{k-1}}b_k$ ($k\ge 2$) to obtain

$$1+k\left(\frac{G_k}{G_{k-1}}b_k-1\right)\le\left(\frac{G_k}{G_{k-1}}b_k\right)^k=\frac{a_k}{G_{k-1}}b_k^k,$$

which can be written as

$$G_{k-1}+k\left(G_kb_k-G_{k-1}\right)\le a_kb_k^k. \tag{2.12}$$

We use (2.12) with $k=n$ to get

$$G_{n-1}+n\left(G_nb_n-G_{n-1}\right)\le a_nb_n^n$$

i.e.

$$nG_n+n\left(b_n-1\right)G_n-a_nb_n^n\le\left(n-1\right)G_{n-1}.$$

In the same way we have, by using (2.12) with $k=n-1,n-2,\dots,2$,

$$\begin{array}{rcl}\left(n-1\right)G_{n-1}+\left(n-1\right)\left(b_{n-1}-1\right)G_{n-1}-a_{n-1}b_{n-1}^{n-1} & \le & \left(n-2\right)G_{n-2}\\ & \vdots & \\ 2G_2+2\left(b_2-1\right)G_2-a_2b_2^2 & \le & G_1.\end{array}$$

Obviously $G_1=a_1$ so that $G_1+\left(b_1-1\right)G_1-a_1b_1=0$ and the inequality (2.10) follows by summing the inequalities above.

□

Remark 2.7. The proof above is somewhat more complicated than the other proofs but it leads to a better result. In fact, this method of proving inequalities uses a well-known principle which is sometimes referred to as Redheffer's recursion principle (see [48]). This principle can also be used to improve several other classical inequalities.

Let $a^{(n)} = \{a_1, a_2, \dots, a_n\}$ be a positive sequence $(n = 1, 2, \dots)$. We define the powermeans $M_{r,n}$ of $a^{(n)}$ in the following way:

$$M_{r,n} = M_{r,n}\left(a^{(n)}\right) = \begin{cases} \left(\frac{1}{n}\sum\limits_{k=1}^{n} a_k^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \left(\prod\limits_{k=1}^{n} a_k\right)^{\frac{1}{n}}, & r = 0. \end{cases}$$

Note that $A_n = M_{1,n}$, $G_n = M_{0,n}$ and $H_n = M_{-1,n}$ are the usual arithmetic, geometric and harmonic means, respectively. We also look at the following sequence of powermeans:

$$M^{r,n} = (M_{r,1}, M_{r,2}, \dots, M_{r,n}).$$

In 1996 B. Mond and J. Pečarić proved the following interesting inequality (between iterative powermeans), (see [38]):

$$M_{s,n}\left(M^{r,n}\right) \leq M_{r,n}\left(M^{s,n}\right), \tag{2.13}$$

for every $r \leq s$. We have equality if and only if $a_1 = \cdots = a_n$. The next proof is based on this result.

Proof 7. We use (2.13) with $s = 1$ and $r = 0$ to obtain

$$\frac{1}{n}\sum_{k=1}^{n} G_k \leq \left(\prod_{k=1}^{n}\left(\frac{1}{k}\sum_{i=1}^{k} a_i\right)\right)^{\frac{1}{n}}. \tag{2.14}$$

By using this inequality and the fact that

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{n} a_i\,,\ k \leq n,$$

we find that

$$\sum_{k=1}^{n} G_k \leq \frac{n}{\sqrt[n]{n!}}\sum_{k=1}^{n} a_k. \tag{2.15}$$

We use our previous estimate (2.2) with $k = n-1$ and get

$$\frac{n^n}{n!} = \frac{n^{n-1}}{(n-1)!} < e^{n-1}\text{, i.e., } \frac{n}{\sqrt[n]{n!}} < e^{1-\frac{1}{n}}.$$

By combining this inequality with (2.14) we have

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k} a_i\right)^{\frac{1}{k}} < e^{1-\frac{1}{n}}\sum_{k=1}^{n} a_k. \tag{2.16}$$

The non-strict inequality (1.1) follows when we let $n \to \infty$. The fact that the inequality actually is strict follows from the fact that equality in (2.15) only can occur when all a_i are equal, but this cannot be true under our assumption that $\sum_{k=1}^{\infty} a_k$ is convergent.

□

Remark 2.8. More information about how (2.13) can be used to prove and improve inequalities can be found in the fairly new papers [11] and [12].

Remark 2.9. We note that if we, in the proof above, combine (2.14) with the following variant of the AG-inequality

$$\begin{aligned}\left(\prod_{k=1}^{n}\frac{1}{k}\sum_{i=1}^{k}a_i\right)^{\frac{1}{n}} &= \left(\frac{1}{n!}\right)^{\frac{1}{n}}\left(a_1\left(a_1+a_2\right)\cdots\left(a_1+a_2+\cdots+a_n\right)\right)^{\frac{1}{n}} \\ &\leq \left(\frac{1}{n!}\right)^{\frac{1}{n}}\frac{\left(na_1+\left(n-1\right)a_2+\cdots+a_n\right)}{n},\end{aligned}$$

we obtain the following strict improvement of (2.16):

$$\sum_{k=1}^{n}\left(\prod_{i=1}^{k}a_i\right)^{\frac{1}{k}} < e^{1-\frac{1}{n}}\sum_{k=1}^{n}\left(1-\frac{k-1}{n}\right)a_k.$$

3. PÓLYA-KNOPP'S INEQUALITY (1.2)

We begin by proving that (1.2) implies (1.1). As before we note that it is enough to prove (1.1) when $\{a_k\}_1^\infty$ is a non-increasing sequence. Use (1.2) with the function $f(x) = a_k$, $x \in [k-1, k)$, $k = 1, 2, \ldots$. Then

$$\int_0^\infty f(x)dx = \sum_{k=1}^{\infty}a_k \tag{3.1}$$

and

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx = \sum_{k=1}^{\infty}\int_{k-1}^{k}\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx. \tag{3.2}$$

Furthermore, it yields that

$$\int_0^1 \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx = a_1 \tag{3.3}$$

and, for $k = 1, 2, \ldots$,

$$\begin{aligned}\int_{k-1}^{k}\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx &= \int_{k-1}^{k}\exp\left(\frac{1}{x}\sum_{i=1}^{k-1}\ln a_i + \frac{x-(k-1)}{x}\ln a_{k+1})\right)dx \\ &\geq \int_{k-1}^{k}\exp\left(\frac{1}{k}\sum_{i=1}^{k}\ln a_i\right)dx = \left(\prod_{i=1}^{k}a_i\right)^{\frac{1}{k}}.\end{aligned} \tag{3.4}$$

The crucial inequality in (3.4) depends on the fact that the integrand is a weighted arithmetic mean of the numbers $\ln a_i$, $i = 1, 2, \ldots, k$, with weights $\frac{1}{x}, \ldots, \frac{1}{x}$ ($k-1$ weights) and $\frac{x-(k-1)}{k}$. Here $k-1 \leq x \leq k$ and since the sequence is decreasing the mean value is smallest for $x = k$ i.e., when all weights $= \frac{1}{k}$. Now (1.1) follows by combining (3.1) - (3.4).

We now present some simple proofs of (1.2) (and thereby some more proofs of (1.1)).

Proof 8. We note that the function $m(x) = -\int_0^x \ln f^*(t)\,dt$ fulfils the conditions to use Carleson's inequality (2.9) (here $f^*(t)$ is the decreasing rearrangement of the function f). Therefore, according to (2.9), it holds that

$$\int_0^\infty x^p \exp\left(\frac{1}{x}\int_0^x \ln f^*(t)\,dt\right)dx \le e^{p+1}\int_0^\infty x^p f^*(x)\,dx, \tag{3.5}$$

for every $p > -1$. Carleson's argument shows that we in fact have strict inequality in (3.5) and especially for $p = 0$ we thereby get Pólya-Knopp's inequality (1.2).

□

Remark 3.1. Carleson did not note this fact explicitly in his paper [10] since he was obviously only interested in giving a simple proof of the inequality (1.1).

We now present two other proofs of (1.2) and thereby of (1.1) which, like Carleson's proof, only are based on convexity arguments.

Proof 9. First we note that

$$\begin{aligned}\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) &= \exp\left(\frac{1}{x}\int_0^x \ln tf(t)dt - \frac{1}{x}\int_0^x \ln tdt\right)\\ &= \exp\left(\frac{1}{x}\int_0^x \ln tf(t)dt\right)\exp\left(-\frac{1}{x}\int_0^x \ln tdt\right).\end{aligned} \tag{3.6}$$

Furthermore, it yields that

$$-\frac{1}{x}\int_0^x \ln tdt = -\frac{1}{x}\left[t\ln t - t\right]_0^x = -\ln x + 1 \tag{3.7}$$

and, in view of Jensen's inequality (or the AG-inequality),

$$\exp\left(\frac{1}{x}\int_0^x \ln tf(t)dt\right) \le \frac{1}{x}\int_0^x tf(t)dt. \tag{3.8}$$

We integrate, use (3.6) – (3.8), change the order of integration and find that

$$\begin{aligned}\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx &\le \int_0^\infty e^{-\ln x+1}\frac{1}{x}\left(\int_0^x tf(t)dt\right)dx\\ &= e\int_0^\infty \frac{1}{x^2}\left(\int_0^x tf(t)dt\right)dx\\ &= e\int_0^\infty tf(t)\int_t^\infty \frac{1}{x^2}dx\\ &= e\int_0^\infty f(t)dt.\end{aligned}$$

The strict inequality follows since equality in Jensen's inequality requires that $tf(t)$ is constant a.e. but this cannot occur since $\int_0^\infty f(x)dx$ is convergent.

□

Remark 3.2. The proof above is partly related to Knopp's original idea (see [32, p. 211]). However, Knopp worked with the interval $[1, x]$ instead of $[0, x]$ and hence Jensen's inequality can not be used. Moreover, Knopp never wrote out the inequality (1.2) explicitly even if it is sometimes referred in the literature as this is the case, see for example [23, p. 250] and [37, p. 143].

Remark 3.3. By modifying the proof above we can easily prove even some weighted versions of (1.2), for instance the following

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) x^p dx < \frac{e}{1-p}\int_0^\infty f(x)x^p dx$$

for every $p < 1$ which is more general than (1.2) and also than (3.5) for $p < 0$.

Proof 10. We first note that if we replace $f(t)$ by $f(t)/t$ in (1.2), then the left hand side in (1.2) equals

$$\int_0^\infty \exp\left(\frac{1}{x}\left(\int_0^x \ln f(t)dt - \int_0^x \ln t dt\right)\right) dx = e\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\frac{dx}{x}$$

since

$$-\frac{1}{x}\int_0^x \ln t dt = -\frac{1}{x}\left[t\ln t - t\right]_0^x = -\ln x + 1.$$

Thus, (1.2) can be written in the equivalent and maybe more natural form

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\frac{dx}{x} < 1\int_0^\infty f(x)\frac{dx}{x}. \tag{3.9}$$

In order to prove (3.9) we use the fact that the function $f(u) = e^u$ is convex and Jensen's inequality:

$$\begin{aligned}\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\frac{dx}{x} &\leq \int_0^\infty \frac{1}{x^2}\left(\int_0^x f(t)dt\right)dx \\ &= \int_0^\infty f(t)\left(\int_t^\infty \frac{1}{x^2}dx\right)dt \\ &= \int_0^\infty f(t)\frac{dt}{t}.\end{aligned}$$

The strict inequality follows in the same way as in Proof 9.

□

4. FURTHER RESULTS AND REMARKS

Remark 4.1. Proof 9 is of course similar to Proof 10 but it contains the important information that (1.1) can be equivalently rewritten in the form (3.9) with the best constant 1. By using this observation and modifying the proof, we find that, in fact, the following more general theorem holds (cf. [29, Theorem 4.1]):

Theorem 4.2. *Let ϕ be a positive and convex function on the range of the measurable function f. Then*

$$\int_0^\infty \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} < \int_0^\infty \phi\left(f(x)\right)\frac{dx}{x}. \tag{4.1}$$

Remark 4.3. By choosing $\phi(u) = e^u$ and replacing $f(x)$ with $\ln f(x)$ we see that (4.1) becomes (3.9) and thereby the equivalent inequality (1.1) and by choosing $\phi(u) = u^p$ we find that (4.1) implies Hardy's inequality in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p \frac{dx}{x} < \int_0^\infty f^p(x)\frac{dx}{x}, p \geq 1, \tag{4.2}$$

which for the case $p > 1$ (after some substitutions) can be written in the usual form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(x)dx,\ p > 1, \tag{4.3}$$

where $g(x) = f\left(x^{1-\frac{1}{p}}\right)x^{-\frac{1}{p}}$. Note especially that Hardy's inequality written in the form (4.2) holds even when $p = 1$ but that the inequality (4.3) then has no meaning.

Remark 4.4. This result and proof can be found in the relatively new paper [29]. We note that the same proof shows that also the more general inequality

$$\int_0^b \phi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{dx}{x} \le \int_0^b \phi\left(f(x)\right)\left(1-\frac{x}{b}\right)\frac{dx}{x}$$

holds for every positive and convex function ϕ on the range of the measurable function ϕ and $0 < b \le \infty$. Especially, this means that if we argue as in Remark 4.3, we get the following improvement of the Pólya-Knopp and Hardy inequalities for finite intervals $(0, b)$, $b < \infty$:

$$\int_0^b \exp\left(\frac{1}{x}\int_0^x f(t)dt\right)dx \le e\int_0^b \left(1-\frac{x}{b}\right)f(x)\,dx$$

respectively

$$\int_0^{b_0}\left(\frac{1}{x}\int_0^x g(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^{b_0}\left(1-\left(\frac{x}{b_0}\right)^{\frac{p-1}{p}}\right)g(x)dx,$$

where $b_0 = b^{p/(p-1)}$ and $g(x) = f\left(x^{(p-1)/p}\right)x^{-\frac{1}{p}}$ as before. These inequalities have recently been proved in the paper [11] (see also [12]) with a different method which is built on the inequalities between mixed means (cf. our Proof 7). The idea in this remark is further developed and applied in [13].

Another interesting question which has recently been studied is to find general weighted versions of the inequality (1.2). Partly guided by the development concerning Hardy type inequalities (see for example the books [33] and [42]) one has asked:
Let $0 < p, q < \infty$. Find necessary and sufficient conditions for the weights (i.e. the positive and measurable functions) $u(x)$ and $v(x)$ so that

$$\left(\int_0^\infty \left(\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)\right)^q u(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{4.4}$$

holds with a stable estimate of the operator norm (= the smallest constant C so that (4.4) holds).
The following has recently been proved:

Theorem 4.5. *Let $0 < p \le q < \infty$. Then the inequality (4.4) holds if and only if*

$$\mathbb{D} := \sup_{x>0} x^{-\frac{1}{p}}\left(\int_0^x w(s)ds\right)^{\frac{1}{q}} < \infty,$$

where

$$w(s) = \left(\exp\left(\frac{1}{s}\int_0^s \ln\frac{1}{v(t)}dt\right)\right)^{\frac{q}{p}} u(s)$$

and

$$\mathbb{D} \le C \le e^{\frac{1}{p}}\mathbb{D}.$$

Theorem 4.6. *Let $0 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then the inequality (4.4) holds if and only if*

$$\mathbb{B} := \left(\int_0^\infty \left(\frac{1}{x} \int_0^x w(x)dx \right)^{\frac{r}{p}} w(x)dx \right)^{\frac{1}{r}},$$

where $w(x)$ is defined as in Theorem 4.5, and $C \approx \mathbb{B}$.

Remark 4.7. These results were recently proved in [45]. We also refer the reader to some earlier results of this type which can be found in the papers [25], [41] and [46]. Further developments of Theorems 4.5 and 4.6 can be found in [40] and the new Ph.D thesis by Maria Nassyrova [39].

We will now present an example of a new generalization of (1.1) (see [29, Theorem 2.1]).

Theorem 4.8. *Let $\{a_k\}_1^\infty$ be a sequence of positive numbers and put $x_i = ia_i \left(1 + \frac{1}{i}\right)^i$, $i = 1, 2, \dots$. Then the following holds for $N \in \mathbb{Z}_+$:*

$$\sum_{k=1}^N G_k + \sum_{k=1}^N \frac{l_k}{k\,(k+1)} \leq \sum_{k=1}^N \left(1 - \frac{k}{N+1}\right) \left(1 + \frac{1}{k}\right)^k a_k, \tag{4.5}$$

where

$$G_k := \sqrt[k]{a_1 a_2 \cdots a_k} \text{ and } l_k := \sum_{i=1}^{[x]} \left(\sqrt{x^*_{k-i+1}} - \sqrt{x^*_i} \right)^2.$$

Here $[x]$ is the usual integer part of x and $\{x^*_k\}$ is the sequence $\{x_k\}$ rearranged in non-increasing order.

Remark 4.9. For previous results of this type we also refer to the papers [2], [3], [4], [44], [54], [56] and the references found there. We note that by using the estimates $l_k \geq 0$, $\left(1 + \frac{1}{k}\right)^k < e$ and letting $N \to \infty$ we get (1.1) as a special case of (4.5).

Remark 4.10. Refinements of Carleman's inequality (1.1) with e replaced by $\left(1 + \frac{1}{k}\right)^k$ have been known since at least 1967 (see [48] and [49] and compare with our Proof 6). We also note that the factor $1 - \frac{k}{N+1}$ in (4.5) means that the "usual" sum on the right hand side of the inequality has been replaced by the equivalent Cesaro sum, i.e., we have calculated the arithmetic mean of partial sums. This mean value is of course strictly less than the "usual" sum since the terms are positive.

Remark 4.11. In the paper [54], P. Yan and G. Sun proved Carleman's inequality (1.1) can be improved in the following way:

$$\sum_{k=1}^\infty \left(\prod_{i=1}^k a_i \right)^{\frac{1}{k}} < e \sum_{k=1}^\infty \left(1 + \frac{1}{k + \frac{1}{5}} \right)^{-\frac{1}{2}} a_k. \tag{4.6}$$

This result easily follows from (4.5) by estimating the important factor $\left(1 + \frac{1}{k}\right)^k$ in the following way:

$$\left(1 + \frac{1}{k}\right)^k \leq e \left(1 + \frac{1}{k + c^*}\right)^{-\frac{1}{2}}, \tag{4.7}$$

where $c^* = \frac{8-e^2}{e^2-4} \approx 0,\ 1802696 < \frac{1}{5}$. The inequality (4.7) does not hold for numbers smaller than c^*. (See [29, Remark 12]). This means that by using (4.5) we see that (4.6) actually can be replaced with the sharper inequality

$$\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}a_i\right)^{\frac{1}{k}}+\sum_{k=1}^{\infty}\frac{l_k}{k\,(k+1)}<e\sum_{k=1}^{\infty}\left(1+\frac{1}{k+c^*}\right)^{-\frac{1}{2}}a_k.$$

Remark 4.12. The factor $\left(1+\frac{1}{k}\right)^k$ has also been of interest in some other new papers. For example M. Gyllenberg and P. Yan recently proved in the paper [18] that

$$\left(1+\frac{1}{k}\right)^k=e\left(1-\sum_{n=1}^{\infty}\frac{a_n}{(1+k)^n}\right),$$

where all a_n are positive and can be calculated recursively. For example $a_1=\frac{1}{2},\ a_2=\frac{1}{24},\ a_3=\frac{1}{48}$ etc. This answers an earlier question raised by Yang (see [56]).

Remark 4.13. We have noted before that Carleson's inequality (2.9) gives both (1.1) and (1.2) as special cases. Another inequality with that property has recently been proved, namely the following (see [29, Theorem 3.1]):

$$\int_0^B\exp\left\{\frac{1}{M(x)}\int_0^x\ln f(t)dM(t)\right\}dM(x)+e\int_0^B\left(1-\frac{M_*(x)}{M(x)}\right)f(x)dM(x)$$
$$\leq e\int_0^B\left(1-\frac{M_*(x)}{M(B)}\right)f(x)dM(x).$$

Here $B\in\mathbb{R}_+,\ M(x)$ is a right continuous and increasing function on $(0,\infty)$ and $M_*(x)$ is a special defined function with the property that $M_*(x)\leq M(x)$. By using this theorem with $M(x)=x$ and $B=\infty$ we get (1.2) and by using it with

$$M(x)=\begin{cases}\frac{1}{2}, & 0\leq x\leq 1,\\ k, & k\leq x\leq k+1,\ k=1,2,\dots\end{cases}$$

we get a refinement of (1.1).

In view of the questions raised in connection to (4.4) it is natural to ask the following which is connected to (1.1): Let $0<p,q<\infty$. Find necessary and sufficient conditions on the non-negative sequences $\{b_k\}_1^\infty$ and $\{d_k\}_1^\infty$ such that

$$\left(\sum_{k=1}^{\infty}\left(\sqrt[k]{a_1a_2\cdots a_k}\right)^q b_k\right)^{\frac{1}{q}}\leq C\left(\sum_{k=1}^{\infty}a_k^p d_k\right)^{\frac{1}{p}} \tag{4.8}$$

holds.

We have the following generalized weighted Carleman's inequality:

Theorem 4.14. *For $k=1,2,\dots$, let $a_k\geq 0, b_k\geq 0$ and $d_k>0$. If $0<p\leq q<\infty$, then the inequality (4.8) holds for some finite constant $C>0$, if and only if*

$$B_1=\sup_{N>0}N^{-\frac{1}{p}}\left(\sum_{k=1}^{N+1}\left(\prod_{i=1}^{k}d_i\right)^{-\frac{q}{kp}}b_k\right)^{\frac{1}{q}}<\infty. \tag{4.9}$$

Moreover, for the best constant C in (4.8) it yields that

$$C\approx B_1. \tag{4.10}$$

Proof. Assume that (4.9) holds. Let first $w_1 = 0$, and replace a_k with $\widetilde{a}_k d_k^{-\frac{1}{p}}$ in (4.8). Then (4.8) is equivalent to

$$\left(\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\widetilde{a}_i\right)^{\frac{q}{k}}\left(\prod_{i=1}^{k}d_i\right)^{-\frac{q}{kp}}b_k\right)^{\frac{1}{q}} \le C\left(\sum_{k=1}^{\infty}\widetilde{a}_k^p\right)^{\frac{1}{p}}$$

or, if $w_k = \left(\prod_{i=1}^{k} d_i\right)^{-\frac{q}{kp}} b_k$,

$$I^{\frac{1}{q}} := \left(\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\widetilde{a}_i\right)^{\frac{q}{k}} w_k\right)^{\frac{1}{q}} \le C\left(\sum_{k=1}^{\infty}\widetilde{a}_k^p\right)^{\frac{1}{p}}. \tag{4.11}$$

Now if $\{a_k^*\}_{k=1}^{\infty}$is the decreasing arrangement of $\{\widetilde{a}_k\}_{k=1}^{\infty}$, then

$$\left(\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}\widetilde{a}_i\right)^{\frac{q}{k}} w_k\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}a_i^*\right)^{\frac{q}{k}} w_k\right)^{\frac{1}{q}}.$$

Let

$$f^*(x) = a_k^* \text{ and } w(x) = w_k \text{ for } x \in [k-1, k)\,.$$

Then

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}\left(\prod_{i=1}^{k}a_i^*\right)^{\frac{q}{k}} w_k\right)^{\frac{1}{q}} \qquad (4.12)\\
&= \sum_{k=1}^{\infty}\int_{k-1}^{k}\left[\exp\left(\sum_{i=1}^{k}\log a_i^{*\frac{1}{k}}\right)\right]^q w_k dx\\
&= \sum_{k=1}^{\infty}\int_{k-1}^{k}\left[\exp\left(\frac{1}{k}\sum_{i=1}^{k-1}\log a_i^* + \frac{1}{k}\log a_k^*\right)\right]^q w_k dx\\
&\le \sum_{k=1}^{\infty}\int_{k-1}^{k}\left[\exp\left(\frac{1}{x}\sum_{i=1}^{k-1}\log a_i^* + \frac{x-(k-1)}{x}\log a_k^*\right)\right]^q w_k dx\\
&= \sum_{k=1}^{\infty}\int_{k-1}^{k}\left[\exp\left(\frac{1}{x}\int_0^x \ln f^*(t)dt\right)\right]^q w(x)dx\\
&= \left(\int_0^{\infty}\left[\exp\left(\frac{1}{x}\int_0^x \ln f^*(t)dt\right)\right]^q w(x)dx\right)^{\frac{1}{q}}.
\end{aligned}$$

Moreover, it follows from Theorem 4.5 that if

$$\mathbb{D} = \sup_{x>0} x^{-\frac{1}{p}}\left(\int_0^x w(t)dt\right)^{\frac{1}{q}} < \infty, \tag{4.13}$$

then the inequality

$$\left(\int_0^{\infty}\left(\exp\left(\frac{1}{x}\int_0^x \ln f^*(t)dt\right)\right)^q w\left(x\right)dx\right)^{\frac{1}{q}} \le C\left(\int_0^{\infty} f^{*p}(x)dx\right)^{\frac{1}{p}} \tag{4.14}$$

holds, and if C is the best possible constant in (4.14), then

$$C \leq e^{\frac{1}{p}} \mathbb{D}. \tag{4.15}$$

Hence, by combining (4.12) with (4.14), we have that

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \left(\sqrt[k]{\widetilde{a}_1 \widetilde{a}_2 \cdots \widetilde{a}_k} \right)^q w_k \right)^{\frac{1}{q}} &\leq C \left(\int_0^{\infty} f^{*p}(x) dx \right)^{\frac{1}{p}} \\ &= C \left(\sum_{k=1}^{\infty} a_k^{*p} \right)^{\frac{1}{p}} = C \left(\sum_{k=1}^{\infty} \widetilde{a}_k^p \right)^{\frac{1}{p}}, \end{aligned} \tag{4.16}$$

i.e. (4.11) (and thus (4.8)) holds whenever (4.13) holds. For $N \in \mathbb{Z}_+$, we have

$$\sup_{N < x \leq N+1} x^{-\frac{1}{p}} \left(\int_0^x w(t) dt \right)^{\frac{1}{q}} \leq N^{-\frac{1}{p}} \left(\sum_{k=1}^{N+1} w_k \right)^{\frac{1}{q}}.$$

Hence

$$\sup_{x>0} x^{-\frac{1}{p}} \left(\int_0^x w(t) dt \right)^{\frac{1}{q}} \leq \sup_{N>0} N^{-\frac{1}{p}} \left(\sum_{k=1}^{N+1} \left(\prod_{i=1}^{k} d_i \right)^{-\frac{q}{kp}} b_k \right)^{\frac{1}{q}} = B_1. \tag{4.17}$$

If $w_1 \neq 0$, then, by using what we just have proved and an elementary inequality, we have

$$I^{\frac{1}{q}} = \left(\widetilde{a}_1^q w_1 + \sum_{k=2}^{\infty} \left(\prod_{i=1}^{k} \widetilde{a}_i \right)^{\frac{q}{k}} w_k \right)^{\frac{1}{q}} \leq \max\left(1, 2^{\frac{1}{q}-1}\right) \left(w_1^{\frac{1}{q}} + C \right) \left(\sum_{k=1}^{\infty} \widetilde{a}_k^p \right)^{\frac{1}{p}}. \tag{4.18}$$

Therefore, by using (4.9), (4.17), (4.13) and (4.18), we conclude that (4.8) holds, and also that the upper estimate holds in (4.10) (when $b_1 = 0$).

On the contrary, assume that (4.8) (and, thus, (4.11)) holds for all non-negative sequences. In particular, let $\widetilde{a}_k = 1$, $k = 1, \ldots, N+1$ and $\widetilde{a}_k = 0$, $k > N+1$. Then the left hand side in (4.11) can be estimated as follows:

$$\left(\sum_{k=1}^{\infty} \left(\prod_{i=1}^{k} \widetilde{a}_i \right)^{\frac{q}{k}} w_k \right)^{\frac{1}{q}} \geq \left(\sum_{k=1}^{N+1} \left(\prod_{i=1}^{k} \widetilde{a}_i \right)^{\frac{q}{k}} w_k \right)^{\frac{1}{q}} = \left(\sum_{k=1}^{N+1} w_k \right)^{\frac{1}{q}}.$$

For the right hand side we have

$$\left(\sum_{k=1}^{\infty} \widetilde{a}_k^p \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{N+1} 1 \right)^{\frac{1}{p}} = (N+1)^{\frac{1}{p}} \leq (2N)^{\frac{1}{p}}$$

and in view of (4.11), it follows that

$$(2N)^{-\frac{1}{p}} \left(\sum_{k=1}^{N+1} w_k \right)^{\frac{1}{q}} \leq C,$$

so that (4.9) and the lower estimate in (4.10) holds. The proof is complete. □

Remark 4.15. We note that our proof gives concrete values of the equivalence constants in (4.10). For example, we always have $2^{-\frac{1}{p}} B_1 \leq C$ and, if, in addition, $b_1 = 0$, then

$$2^{-\frac{1}{p}} B_1 \leq C \leq e^{\frac{1}{p}} B_1.$$

Remark 4.16. In [25] the weighted Carleman's inequality (4.8) was proved with another condition than B_1 and without any estimate of the best constant C.

5. FINAL REMARKS ABOUT TORSTEN CARLEMAN AND HIS WORK

Remark 5.1. A main reference concerning Torsten Carleman and his mathematics is of course the book [19] of L. Gårding (see pp. 233–276). In this book Carleman is described in the following way: "With Torsten Carleman (1892-1949) Sweden got their so far most outstanding mathematician." It is therefore not curious that Gårding spent the next 30 pages to describe Carleman and his mathematical work and no other mathematician was given even close to so much space in the book. It is remarkable that (1.1) is not explicitly mentioned in Gårding's book, which can depend on the fact that he (as well as Carleman himself) obviously regarded the inequality only as a necessary tool to prove his important main results concerning quasianalytical functions. However, as we have seen in this article, Carleman's inequality (1.1) and its continuous variant (Polya-Knopp's inequality (1.2)) has attracted a lot of attention and it is even mentioned in the title of a number of papers. See our list of references containing 58 references, Chapter 4 in the book [37] (with 174 references), Chapter 1 in the book [33] and the recently published review paper [44] (with 53 references). And the interest seems only to have increased during the last few years.

Remark 5.2. (About the person Torsten Carleman). There is a lot of interesting information in Gårding's book [19] and some complementary information can be found in [58]. Tage Gillis Torsten Carleman was born 8 July, 1892. He defended his Ph.D. thesis 1917 at Uppsala University. In 1923 he was appointed a full professor at the Lund University. Shortly after this, and on an initiative of Professor Gösta Mittag-Leffler (which has initiated and given name of the famous mathematical research institute in Djursholm, Sweden), he was called as professor at Stockholms University, in 1924. He died in 1949. Carleman was a remarkable person and there are many rumours concerning him (see e.g. Professor Bo Kjellberg's interesting and very personal description in [31, p. 93]).

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