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# THE BEST CONSTANT FOR A GEOMETRIC INEQUALITY

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ABSTRACT. In this paper, we prove that the best constant for the geometric inequality $\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is a root of one polynomial by the method of mathematical analysis and linear algebra.

*Key words and phrases:* Best Constant, Geometric Inequality, Euler's Inequality, Gerretsen's Inequality, Sylvester's Resultant.

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## 1. INTRODUCTION AND MAIN RESULTS

In 1993, Shi-Chang Shi strengthened the familiar geometric inequality (in triangle)

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{\sqrt{3}}{2r} \tag{1.1}$$

to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left(\frac{1}{r}+\frac{1}{R}\right) \tag{1.2}$$

in [1]. After several months, Ji Chen obtained the following beautiful and strong inequality chain in [2].

$$\frac{11\sqrt{3}}{5R+12r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right). \tag{1.3}$$

In the same year, Xi-Ling Huang posed the following interesting inequality problem in [3].

**Problem 1.** Determine the best constant $k$ for which the inequality below holds

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le \frac{1}{\sqrt{3}}\left[\frac{1}{R}+\frac{1}{r}+\frac{1}{k}\left(\frac{2}{R}-\frac{1}{r}\right)\right]. \tag{1.4}$$

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In 1996, Sheng-Li Chen solved Problem 1 completely in [4]. He obtained the following theorem.

**Theorem 1.1.** *The best constant* $k$ *for the inequality* (1.2) *is* $2(1+\sqrt[3]{2}+\sqrt[3]{4})$.

In the same year, Xue-Zhi Yang [5] strengthened the inequality

$$\frac{11\sqrt{3}}{5R+12r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \tag{1.5}$$

to

$$\frac{243\sqrt{3}}{110R+266r} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}. \tag{1.6}$$

In this paper we will determine the best constant for the inequality

$$\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}, \tag{1.7}$$

where $0<k<5$. We obtain the following theorem.

**Theorem 1.2.** *The maximum value of* $k$ *for which the inequality* (1.7) *holds is the root on the open interval* $\left(0,\frac{1}{15}\right)$ *of the following equation*

$$405k^5+6705k^4+129586k^3+1050976k^2+2795373k-62181=0.$$

*Its approximation is* $0.02206078402$*.*

In fact, let $k=\frac{5}{243}\approx 0.020576131687<0.02206078402$, we immediately find that the inequality (1.7) is just the inequality (1.6).

## 2. LEMMAS

In order to prove Theorem 1.1, we require several lemmas. The second was obtained by Sheng-Li Chen in [6] (see also [7]).

**Lemma 2.1.** *If* $0<k<5$*, then the inequality*

$$\frac{11\sqrt{3}}{5R+12r+k(2r-R)} \le \frac{1}{\sqrt{3}}\left(\frac{5}{4R}+\frac{7}{8r}\right) \tag{2.1}$$

*holds if and only if* $0<k\le\frac{5}{12}$*.*

*Proof.* Since $0<k<5$, it is obvious that $5R+12r+k(2r-R)>0$. Therefore, (2.1) is equivalent to

$$7(5-k)R^2+(4k-130)Rr+(20k+120)r^2\ge 0. \tag{2.2}$$

Setting $\frac{R}{2r}=x$, then with Euler's Inequality $R\ge 2r$, we have $x\ge 1$. Inequality (2.2) is equivalent to

$$28(5-k)x^2+2(4k-130)x+20k+120\ge 0,$$

that is,

$$4(x-1)[(35-7k)x-5k-30]\ge 0. \tag{2.3}$$

Considering that $x\ge 1$, (2.3) holds if and only if $(35-7k)x-5k-30\ge 0 (x\ge 1)$. Namely, $k\le\frac{5(7x-6)}{7x+5}$ or $k\le \min\frac{5(7x-6)}{7x+5}(x\ge 1)$.
Define the function

$$f(x)=\frac{5(7x-6)}{7x+5}(x\ge 1).$$

Calculating the derivative for $f(x)$, we get

$$f'(x) = -\frac{35(7x-6)}{(7x+5)^2} + \frac{35}{7x+5} = \frac{385}{(7x+5)^2} > 0,$$

and thus the function $f(x)$ is strictly monotone increasing on the interval $[1, +\infty)$. Then $f(x) \geq f(1) = \frac{5}{12}$. That is $\min f(x) = 1$ for $x \geq 1$. So $k \leq \frac{5}{12}$, combining $0 < k < 5$, we immediately obtain $0 < k \leq \frac{5}{12}$. Thus, Lemma 2.1 is proved. □

**Lemma 2.2.** [6] *The homogeneous inequality $F(R, r, s) \geq (>)0$ in triangle which form is equivalent to $p \geq (>)f(R, r)$ holds if and only if it holds by setting $R = 2$, $r = 1 - x^2$, $p = \sqrt{(1-x)(3+x)^3}$, where $0 \leq x < 1$. And the form which is equivalent to $p \leq (<)f(R, r)$ holds if and only if it holds by setting the same substitution, where $-1 < x \leq 0$.*

*Proof.* It is well known that the following two inequalities

$$p^2 \geq 2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{r(R - 2r)} \tag{2.4}$$

and

$$p^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{r(R - 2r)} \tag{2.5}$$

hold in any triangle $ABC$.
Now we prove the inequality (2.4) with equality holding if and only if $\Delta ABC$ is an isosceles triangle whose top-angle is greater than or equal to $60°$, and the inequality (2.5) with equality holding if and only if $\Delta ABC$ is an isosceles triangle whose top-angle is less than or equal to $60°$.
Let $A$ be the top- angle of isosceles triangle $ABC$, and let

$$t = \sin\frac{A}{2} (= \cos B = \cos C) \in (0, 1),$$

then

$$\sin\frac{B}{2} = \sin\frac{C}{2} = \sqrt{\frac{1-t}{2}}.$$

With known identities

$$r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, \qquad p = R(\sin A + \sin B + \sin C).$$

in triangle, we easily obtain

$$r = 2Rt(1-t), \qquad p = 2R(1+t)\sqrt{1-t^2}. \tag{2.6}$$

We put the identities (2.6) into the inequality (2.4) and (2.5), with simple calculations, we can find the inequality (2.4) with equality holding if and only if $t \in \left[\frac{1}{2}, 1\right)$ or $A \geq 60°$; the inequality (2.5) with equality holding if and only if $t \in \left(0, \frac{1}{2}\right]$ or $A \leq 60°$.
Then we prove the following two propositions.

**Proposition 2.3.** *For every triangle $ABC$, there are isosceles triangle $A_1B_1C_1$ with top angle $A_1 \geq 60°$ and isosceles triangle $A_2B_2C_2$ with top angle $A_1 \leq 60°$ make*

$$R_1 = R_2 = R, \quad r_1 = r_2 = r; \quad p_1 \leq p \leq p_2,$$

*with $p = p_1$ holding if and only if $\Delta ABC$ is an isosceles triangle with top angle $A \geq 60°$, $p = p_2$ holding if and only if $\Delta ABC$ is an isosceles triangle with top angle $A \leq 60°$.*

*Proof.* Denote $\odot O$ as the circumcircle of $\Delta ABC$, then there are inscribed isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ of $\odot O$ which satisfy the next two identities:

$$\frac{A_1}{2} = \arcsin \frac{1}{2}\left(1+\sqrt{1-\frac{2r}{R}}\right),$$
$$\frac{A_2}{2} = \arcsin \frac{1}{2}\left(1-\sqrt{1-\frac{2r}{R}}\right).$$

Then $A_1 \geq 60^\circ$, $A_2 \leq 60^\circ$ and

$$\sin\frac{A_1}{2}\left(1-\sin\frac{A_1}{2}\right) = \frac{r}{2R}, \tag{2.7}$$

$$\sin\frac{A_2}{2}\left(1-\sin\frac{A_2}{2}\right) = \frac{r}{2R}. \tag{2.8}$$

For isosceles triangles $A_1B_1C_1$ where the top-angle is $A_1$ and $A_2B_2C_2$ where the top-angle is $A_2$, we have

$$\sin\frac{A_1}{2}\left(1-\sin\frac{A_1}{2}\right) = \frac{r_1}{2R_1}, \tag{2.9}$$

$$\sin\frac{A_2}{2}\left(1-\sin\frac{A_2}{2}\right) = \frac{r_2}{2R_2}. \tag{2.10}$$

From (2.7) to (2.10), we get $\frac{r}{R} = \frac{r_1}{R_1} = \frac{r_2}{R_2}$, and it is easy to see that $R = R_1 = R_2$, so $r_1 = r_2 = r$. Denote $\varphi(R, r)$ to be the right of (2.4), then $p^2 \geq \varphi(R, r) = \varphi(R_1, r_1) = p_1^2$, so $p \geq p_1$. In the same manner, we can prove that $p \leq p_2$. □

**Proposition 2.4.**

(i) *If the inequality* $p \geq (>) f_1(R, r)$ *holds for any isosceles triangle whose top-angle is greater than or equal to* $60^\circ$*, then the inequality* $p \geq (>) f_1(R, r)$ *holds for any triangle.*

(ii) *If the inequality* $p \leq (<) f_1(R, r)$ *holds for any isosceles triangle whose top-angle is less than or equal to* $60^\circ$*, then the inequality* $p \leq (<) f_1(R, r)$ *holds for any triangle.*

*Proof.* For any $\Delta A'B'C'$, with Proposition 2.3, we know there is an isosceles triangle $A_1B_1C_1$ which make

$$R_1 = R', \quad r_1 = r', \quad p_1 \leq p'.$$

Because the inequality $p \geq (>) f_1(R, r)$ holds for isosceles triangle $A_1B_1C_1$, we have

$$p' \geq p_1 \geq (>) f_1(R_1, r_1) = f_1(R', r').$$

Thus, the inequality $p \geq (>) f_1(R, r)$ holds for $\Delta A'B'C'$. In the same way we can prove $(ii)$. □

From Proposition 2.4, the homogeneous inequality in triangle whose form is equivalent to $p \geq (>) f(R, r)$ holds if and only if it holds by setting $R = 2$, $r = 4t(1-t)$, $p = 4(1+t)\sqrt{1-t^2}$. Taking $t = \frac{x+1}{2}$, we immediately get $r = 1 - x^2$, $p = \sqrt{(1-x)(3+x)^3}$, where $0 \leq x < 1$. For the homogeneous inequality in triangle whose form is equivalent to $p \leq (<) f(R, r)$, we only need to change the range of $x$. Namely, we change $0 \leq x < 1$ to be $-1 < x \leq 0$.
Thus, the proof of Lemma 2.2 is completed. (The proof was given by Sheng-Li Chen in [6].) □

**Lemma 2.5.** [8] *Denote*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$
$$g(x) = b_0x^m + b_1x^{m-1} + \cdots + b_m.$$

*If* $a_0 \neq 0$ *or* $b_0 \neq 0$*, then the polynomials* $f(x)$ *and* $g(x)$ *have a common root if and only if*

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0,$$

*where* $R(f,g)$ *is Sylvester's Resultant of* $f(x)$ *and* $g(x)$*.*

## 3. PROOF OF THEOREM 1.1

*Proof.* With known identities $abc = 4Rrp$ and $ab + bc + ca = p^2 + 4Rr + r^2$ in triangle, we easily know the inequality (1.7) is equivalent to

$$\frac{11\sqrt{3}}{5R + 12r + k(2r - R)} \leq \frac{p^2 + 4Rr + r^2}{4Rrp}. \tag{3.1}$$

The inequality (3.1) is equivalent to the following inequality

$$[5R + 12r + k(2r - R)]p^2 - 44\sqrt{3}Rrp + [5R + 12r + k(2r - R)](4Rr + r^2) \geq 0. \tag{3.2}$$

(i) If

$$\Delta(R,r) = (44\sqrt{3}Rr)^2 - 4[5R + 12r + k(2r - R)]^2(4Rr + r^2) < 0,$$

it is obvious that the inequality (3.2) holds.

(ii) If

$$\Delta(R,r) = (44\sqrt{3}Rr)^2 - 4[5R + 12r + k(2r - R)]^2(4Rr + r^2) \geq 0,$$

then the inequality (3.2) is equivalent to

$$p \geq \frac{44\sqrt{3}Rr + \sqrt{\Delta(R,r)}}{2[5R + 12r + k(2r - R)]} \tag{3.3}$$

or

$$p \leq \frac{44\sqrt{3}Rr - \sqrt{\Delta(R,r)}}{2[5R + 12r + k(2r - R)]}. \tag{3.4}$$

In fact, the inequality (3.4) does not hold. From (1.3) and (1.7), we have

$$\frac{11\sqrt{3}}{5R + 12r + k(2r - R)} \leq \frac{1}{\sqrt{3}}\left(\frac{5}{4R} + \frac{7}{8r}\right). \tag{3.5}$$

By Lemma 2.1, we know that $0 < k \leq \frac{5}{12}$. It is easy to see that the following inequalities hold

$$\frac{44\sqrt{3}Rr - \sqrt{\Delta(R,r)}}{2[5R + 12r + k(2r - R)]} \leq \frac{44\sqrt{3}Rr}{2[5R + 12r + k(2r - R)]} \leq \frac{22\sqrt{3}Rr}{\left[5R + 12r + \frac{5}{12}(2r - R)\right]}. \tag{3.6}$$

Now we prove the next inequality

$$p \geq \frac{22\sqrt{3}Rr}{\left[5R + 12r + \frac{5}{12}(2r - R)\right]}. \tag{3.7}$$

The inequality (3.7) is equivalent to

$$p^2 \geq \frac{1452R^2r^2}{\left[5R + 12r + \frac{5}{12}(2r - R)\right]^2}. \tag{3.8}$$

With Gerretsen's Inequality $p^2 \geq 16Rr - 5r^2$, in order to prove the inequality (3.8), we only need to prove the following inequality.

$$16Rr - 5r^2 \geq \frac{1452R^2r^2}{\left[5R + 12r + \frac{5}{12}(2r - R)\right]^2}. \tag{3.9}$$

The inequality (3.9) is equivalent to

$$r(400R^3 + 387R^2 + 2436Rr - 980r^3) \geq 0. \tag{3.10}$$

With Euler's inequality $R \geq 2r$, we easily see that the inequality (3.10) holds. So, the inequality (3.7) holds. Then the inequality (3.4) does not hold. Therefore, the inequality (3.2) is equivalent to the inequality (3.3). From Lemma 2.2, the inequality (3.2) holds if and only if the following inequality holds.

$$8(1 - x)(3 + x)\left[(2x + 3)(11 - 6x^2 - kx^2) - 11\sqrt{3}(x + 1)\sqrt{(1 - x)(3 + x)}\right] \geq 0 \quad (0 \leq x < 1). \tag{3.11}$$

The inequality (3.11) holds when $x = 0$. When $0 < x < 1$, the inequality (3.11) is equivalent to

$$k \leq \frac{(2x + 3)(11 - 6x^2) - 11\sqrt{3}(x + 1)\sqrt{(1 - x)(3 + x)}}{x^2(2x + 3)}. \tag{3.12}$$

Define the function

$$g(x) = \frac{(2x + 3)(11 - 6x^2) - 11\sqrt{3}(x + 1)\sqrt{(1 - x)(3 + x)}}{x^2(2x + 3)}, \quad (0 < x < 1). \tag{3.13}$$

Calculating the derivative for $g(x)$, we get

$$g'(x) = \frac{-22\left[\sqrt{3}(x^4 + 5x^3 + 2x^2 - 9x - 9) + (2x + 3)^2\sqrt{(1 - x)(3 + x)}\right]}{x^3(2x + 3)^2\sqrt{(1 - x)(3 + x)}}. \tag{3.14}$$

Let $g'(x) = 0$, we get

$$3x^5 + 30x^4 + 103x^3 + 134x^2 + 48x - 18 = 0, \quad (0 < x < 1). \tag{3.15}$$

It is easy to see that the equation (3.15) has the only one positive root on the open interval $(0,1)$. Denote $x_0$ to be the root of the equation (3.15). Then

$$g(x)_{\min} = g(x_0) = \frac{(2x_0+3)(11-6x_0^2) - 11\sqrt{3}(x_0+1)\sqrt{(1-x_0)(3+x_0)}}{x_0^2(2x_0+3)}.$$

Therefore, the maximum of $k$ is $g(x_0)$. Now we prove $g(x_0)$ is the root of the equation

$$405k^5 + 6705k^4 + 129586k^3 + 1050976k^2 + 2795373k - 62181 = 0.$$

It is easy to find that $g(x_0)$ is a root of the following equation.

$$x_0^2(2x_0+3)^2t^2 - 2(2x_0+3)^2(11-6x_0^2)t + 144x_0^4 + 432x_0^3 + 159x_0^2 - 132x_0 + 22 = 0.$$

We know that

$$3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0.$$

Considering the simultaneous equations

$$\begin{cases} x_0^2(2x_0+3)^2t^2 - 2(2x_0+3)^2(11-6x_0^2)t + 144x_0^4 \\ \qquad\qquad +432x_0^3 + 159x_0^2 - 132x_0 + 22 = 0 \\ \\ 3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0 \end{cases} \tag{3.16}$$

The simultaneous equations (3.16) can be changed to the simultaneous equations as follows.

$$\begin{cases} 4(t+6)^2x_0^4 + 12(t+6)^2x_0^3 + (9t^2+20t+159)x_0^2 \\ \qquad\qquad -132(2t+1)x_0 - 198t + 22 = 0 \\ \\ 3x_0^5 + 30x_0^4 + 103x_0^3 + 134x_0^2 + 48x_0 - 18 = 0 \end{cases} \tag{3.17}$$

Then,

$$R_{x_0}(f,g) = \begin{vmatrix} 4(t+6)^2 & 12(t+6)^2 & \cdots & 22-198t & 0 & \cdots & 0 \\ 0 & 4(t+6)^2 & \cdots & \cdots & 22-198t & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 4(t+6)^2 & \cdots & \cdots & 22-198t \\ 3 & 30 & \cdots & -18 & 0 & 0 & 0 \\ 0 & 3 & 30 & \cdots & -18 & 0 & 0 \\ 0 & 0 & 3 & 30 & \cdots & -18 & 0 \\ 0 & 0 & 0 & 3 & 30 & \cdots & -18 \end{vmatrix}$$

$$= 100(405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181)$$
$$\times (405t^5 - 178425t^4 - 1656374t^3 - 13317290t^2 - 100675599t - 330639021).$$

The solution of the equation $R_{x_0}(f,g) = 0$ is the union of the solution of the equation

$$405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181 = 0, \tag{3.18}$$

and the equation

$$405t^5 - 178425t^4 - 1656374t^3 - 13317290t^2 - 100675599t - 330639021 = 0. \tag{3.19}$$

With differential calculus, it is easy to see that the equation (3.19) has no root on the interval $[0,1]$. We can get $g(x_0) < 1$, with Lemma 2.5, we can conclude that $g(x_0)$ is the real root of the equation (3.18). Define the function

$$f(t) = 405t^5 + 6705t^4 + 129586t^3 + 1050976t^2 + 2795373t - 62181. \tag{3.20}$$

Then $f(\frac{1}{15}) = \frac{2174963624}{16875} > 0.$ Therefore, the real root of the equation (3.18) is on the interval $(0, \frac{1}{15})$.
Thus, the proof of Theorem 1.2 is completed. □

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