Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 2, Article 35, 2004

# SOME NORMALITY CRITERIA

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*Received 12 October, 2003; accepted 01 January, 2004*
*Communicated by H.M. Srivastava*

ABSTRACT. In the paper we prove some sufficient conditions for a family of meromorphic functions to be normal in a domain.

*Key words and phrases:* Meromorphic function, Normality.

2000 *Mathematics Subject Classification.* Primary 30D45, Secondary 30D35.

## 1. INTRODUCTION AND RESULTS

Let $\mathbb{C}$ be the open complex plane and $\mathcal{D} \subset \mathbb{C}$ be a domain. A family $\mathcal{F}$ of meromorphic functions defined in $\mathcal{D}$ is said to be normal, in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$ there exists a subsequence $f_{n_j}$ such that $f_{n_j}$ converges spherically, locally and uniformly in $\mathcal{D}$ to a meromorphic function or $\infty$.

$\mathcal{F}$ is said to be normal at a point $z_0 \in \mathcal{D}$ if there exists a neighbourhood of $z_0$ in which $\mathcal{F}$ is normal. It is well known that $\mathcal{F}$ is normal in $\mathcal{D}$ if and only if it is normal at every point of $\mathcal{D}$.

It is an interesting problem to find out criteria for normality of a family of analytic or meromorphic functions. In recent years this problem attracted the attention of a number of researchers worldwide.

In 1969 D. Drasin [5] proved the following normality criterion.

**Theorem A.** *Let $\mathcal{F}$ be a family of analytic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, $b$ be two finite numbers. If for every $f \in \mathcal{F}$, $f' - af^n - b$ has no zero then $\mathcal{F}$ is normal, where $n(\geq 3)$ is an integer.*

Chen-Fang [2] and Ye [21] independently proved that *Theorem A* also holds for $n = 2$. A number of authors {cf. [3, 11, 12, 13, 16, 24]} extended *Theorem A* to a family of meromorphic functions in a domain. Their results can be combined in the following theorem.

ISSN (electronic): 1443-5756

144-03

**Theorem B.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, $b$ be two finite numbers. If for every $f \in \mathcal{F}$, $f' - af^n - b$ has no zero then $\mathcal{F}$ is normal, where $n(\geq 3)$ is an integer.*

Li [12], Li [13] and Langley [11] proved *Theorem B* for $n \geq 5$, Pang [16] proved for $n = 4$ and Chen-Fang [3], Zalcman [24] proved for $n = 3$. Fang-Yuan [6] showed that *Theorem B* does not, in general, hold for $n = 2$. For the case $n = 2$ they [6] proved the following result.

**Theorem C.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, $b$ be two finite numbers. If $f' - af^2 - b$ has no zero and $f$ has no simple and double pole for every $f \in \mathcal{F}$ then $\mathcal{F}$ is normal.*

Fang-Yuan [6] mentioned the following example from which it appears that the condition for each $f \in \mathcal{F}$ not to have any simple and double pole is necessary for *Theorem C*.

**Example 1.1.** Let $f_n(z) = nz(z\sqrt{n} - 1)^{-2}$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$. Then each $f_n$ has only a double pole and a simple zero. Also $f_n' + f_n^2 = n(z\sqrt{n} - 1)^{-4} \neq 0$. Since $f_n^{\#}(0) = n \to \infty$ as $n \to \infty$, it follows from Marty's criterion that $\{f_n\}$ is not normal in $\mathcal{D}$.

However, the following example suggests that the restriction on the poles of $f \in \mathcal{F}$ may be relaxed at the cost of some restriction imposed on the zeros of $f \in \mathcal{F}$.

**Example 1.2.** Let $f_n(z) = nz^{-2}$ for $n = 3, 4, \ldots$ and $\mathcal{D} : |z| < 1$. Then each $f_n$ has only a double pole and no simple zero. Also we see that $f_n' + f_n^2 = n(n - 2z)z^{-4} \neq 0$ in $\mathcal{D}$. Since

$$f_n^{\#}(z) = \frac{2n|z|}{|z|^2 + n^2} \leq \frac{2}{n} < 1$$

in $\mathcal{D}$, it follows from Marty's criterion that the family $\{f_n\}$ is normal in $\mathcal{D}$.

Now we state the first theorem of the paper.

**Theorem 1.1.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ such that no $f \in \mathcal{F}$ has any simple zero and simple pole. Let*

$$E_f = \{z : z \in \mathcal{D} \text{ and } f'(z) - af^2(z) = b\},$$

*where $a(\neq 0)$, $b$ are two finite numbers.*

*If there exists a positive number $M$ such that for every $f \in \mathcal{F}$, $|f(z)| \leq M$ whenever $z \in E_f$, then $\mathcal{F}$ is normal.*

The following examples together with *Example 1.1* show that the condition of *Theorem 1.1* on the zeros and poles are necessary.

**Example 1.3.** Let $f_n(z) = n \tan nz$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < \pi$. Then $f_n$ has only simple zeros and simple poles. Also we see that $f_n' - f_n^2 = n^2 \neq 0$. Since $f_n^{\#}(0) = n^2 \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal.

**Example 1.4.** Let $f_n(z) = (1 + e^{2nz})^{-1}$ for $n = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$. Then $f_n$ has no simple zero and no multiple pole. Also we see that $f_n' + f_n^2 \neq 1$. Since $f_n^{\#}(0) = \frac{2n}{3} \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal.

Drasin [18, p. 130] also proved the following normality criterion which involves differential polynomials.

**Theorem D.** *Let $\mathcal{F}$ be a family of analytic functions in a domain $\mathcal{D}$ and $a_0, a_1, \ldots, a_{k-1}$ be finite constants, where $k$ is a positive integer. Let*

$$H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f^{(1)} + a_0 f.$$

*If for every $f \in \mathcal{F}$*

(i) *$f$ has no zero,*
(ii) *$H(f)-1$ has no zero of multiplicity less than $k+2$,*

*then $\mathcal{F}$ is normal.*

Recently Fang-Yuan [6] proved that *Theorem D* remains valid even if $H(f)-1$ has only multiple zeros for every $f \in \mathcal{F}$. In the next theorem we extend *Theorem D* to a family of meromorphic functions which also improves a result of Fang-Yuan [6].

**Theorem 1.2.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and*

$$H(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \ldots + a_1 f^{(1)} + a_0 f,$$

*where $a_0, a_1, \ldots, a_{k-1}$ are finite constants and $k$ is a positive integer.*
*Let*

$$E_f = \{z : z \in \mathcal{D} \text{ and } z \text{ is a simple zero of } H(f)-1\}.$$

*If for every $f \in \mathcal{F}$*

(i) *$f$ has no pole of multiplicity less than $3+k$,*
(ii) *$f$ has no zero,*
(iii) *there exists a positive constant $M$ such that $|f(z)| \geq M$ whenever $z \in E_f$,*

*then $\mathcal{F}$ is normal.*

The following examples show that conditions (ii) and (iii) of *Theorem 1.2* are necessary, leaving the question of necessity of the condition (i) as open.

**Example 1.5.** Let $f_n(z) = nz$ for $n = 2, 3, \ldots$, $\mathcal{D} : |z| < 1$, $H(f) = f' - f$ and $M = \frac{1}{2}$. Then each $f_n$ has a zero at $z = 0$ and $E_{f_n} = \{1 - \frac{1}{n}\}$ for $n = 2, 3, \ldots$. So $|f(1-\frac{1}{n})| = n - 1 \geq M$ for $n = 2, 3, \ldots$. Since $f_n^{\#}(0) = n \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathcal{D}$.

**Example 1.6.** Let $f_n(z) = e^{nz}$ for $n = 2, 3, \ldots$, $\mathcal{D} : |z| < 1$ and $H(f) = f' - f$. Then each $f_n$ has no zero and $E_{f_n} = \{z : z \in \mathcal{D} \text{ and } (n-1)e^{nz} = 1\}$ for $n = 2, 3, \ldots$. Also we see that for $z \in E_{f_n}$, $|f_n(z)| = \frac{1}{n-1} \to 0$ as $n \to \infty$. Since $f_n^{\#}(0) = \frac{n}{2} \to \infty$ as $n \to \infty$, by Marty's criterion the family $\{f_n\}$ is not normal in $\mathcal{D}$.

In connection to *Theorem A* Chen-Fang [3] proposed the following conjecture:

**Conjecture 1.3.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$. If for every function $f \in \mathcal{F}$, $f^{(k)} - af^n - b$ has no zero in $\mathcal{D}$ then $\mathcal{F}$ is normal, where $a(\neq 0)$, $b$ are two finite numbers and $k$, $n(\geq k+2)$ are positive integers.*

In response to this conjecture Xu [23] proved the following result.

**Theorem E.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, $b$ be two finite constants. If $k$ and $n$ are positive integers such that $n \geq k+2$ and for every $f \in \mathcal{F}$*

(i) *$f^{(k)} - af^n - b$ has no zero,*
(ii) *$f$ has no simple pole,*

*then $\mathcal{F}$ is normal.*

The condition (ii) of *Theorem E* can be dropped if we choose $n \geq k+4$ (cf. [15, 17]). Also some improvement of *Theorem E* can be found in [22]. In the next theorem we investigate the situation when the power of $f$ is negative in condition (i) of *Theorem E*.

**Theorem 1.4.** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and $a(\neq 0)$, $b$ be two finite numbers. Suppose that $E_f = \{z : z \in \mathcal{D} \text{ and } f^{(k)}(z) + af^{-n}(z) = b\}$, where $k, n(\geq k)$ are positive integers.*
*If for every $f \in \mathcal{F}$*

(i) *$f$ has no zero of multiplicity less than $k$,*
(ii) *there exists a positive number $M$ such that for every $f \in \mathcal{F}$, $|f(z)| \geq M$ whenever $z \in E_f$,*

*then $\mathcal{F}$ is normal.*

Following examples show that the conditions of *Theorem 1.4* are necessary.

**Example 1.7.** Let $f_p(z) = pz^2$ for $p = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$, $n = k = 3$, $a = 1$, $b = 0$. Then $f_p$ has only a double zero and $E_{f_p} = \emptyset$. Since $f_p(0) = 0$ and for $z \neq 0$, $f_p(z) \to \infty$ as $p \to \infty$, it follows that the family $\{f_p\}$ is not normal.

**Example 1.8.** Let $f_p(z) = pz$ for $p = 1, 2, \ldots$ and $\mathcal{D} : |z| < 1$, $n = k = 1$. Then $f_p$ has simple zero at the origin and for any two finite numbers $a(\neq 0)$, $b$, $E_{f_p} = \{a/p(b-p)\}$ so that $|f_p(z)| \to 0$ as $p \to \infty$ whenever $z \in E_{f_p}$. Since $f_p^{\#}(0) = p \to \infty$ as $p \to \infty$, by Marty's criterion the family $\{f_p\}$ is not normal.

For the standard definitions and notations of the value distribution theory we refer to [8, 18].

## 2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

**Lemma 2.1.** [1] *Let $f$ be a transcendental meromorphic function of finite order in $\mathbb{C}$. If $f$ has no simple zero then $f'$ assumes every non-zero finite value infinitely often.*

**Lemma 2.2.** [10] *Let $f$ be a nonconstant rational function in $\mathbb{C}$ having no simple zero and simple pole. Then $f'$ assumes every non-zero finite value.*

The following lemma can be proved in the line of [9].

**Lemma 2.3.** *Let $f$ be a meromorphic function in $\mathbb{C}$ such that $f^{(k)} \not\equiv 0$. Suppose that $\psi = f^n f^{(k)}$, where $k, n$ are positive integers. If $n > k = 2$ or $n \geq k \geq 3$ then*

$$\left\{1 - \frac{1+k}{n+k} - \frac{n(1+k)}{(n+k)(n+k+1)}\right\} T(r,\psi) \leq \overline{N}(r,a;\psi) + S(r,\psi),$$

*where $a(\neq 0, \infty)$ is a constant.*

**Lemma 2.4.** [19] *Let $f$ be a transcendental meromorphic function in $\mathbb{C}$ and $\psi = f^n f^{(2)}$, where $n(\geq 2)$ is an integer. Then*

$$\limsup_{r\to\infty} \frac{\overline{N}(r,a;\psi)}{T(r,\psi)} > 0,$$

*where $a(\neq 0, \infty)$ is a constant.*

The following lemma is a combination of the results of [3, 7, 14].

**Lemma 2.5.** *Let $f$ be a transcendental meromorphic function in $\mathbb{C}$. Then $f^n f'$ assumes every non-zero finite value infinitely often, where $n(\geq 1)$ is an integer.*

**Lemma 2.6.** *Let $f$ be a non-constant rational function in $\mathbb{C}$. Then $f^n f'$ assumes every non-zero finite value.*

*Proof.* Let $g = f^{n+1}/(n+1)$. Then $g$ is a nonconstant rational function having no simple zero and simple pole. So by *Lemma 2.2* $g' = f^n f'$ assumes every non-zero finite value. This proves the lemma. □

**Lemma 2.7.** *Let $f$ be a rational function in $\mathbb{C}$ such that $f^{(2)} \not\equiv 0$. Then $\psi = f^2 f^{(2)}$ assumes every non-zero finite value.*

*Proof.* Let $f = p/q$, where $p$, $q$ are polynomials of degree $m$, $n$ respectively and $p$, $q$ have no common factor.

Let $a$ be a non-zero finite number. We now consider the following cases.

**Case 1.** *Let $m = n$. Then $f = \alpha + p_1/q$, where $\alpha$ is a constant and $p_1$ is a polynomial of degree $m_1 < n$.*

*Now*

$$f' = \frac{p_1'q - p_1q'}{q^2} = \frac{p_2}{q_2}, \text{ say,}$$

*where $p_2$ and $q_2$ are polynomials of degree $m_2 = m_1 + n - 1$ and $n_2 = 2n$. Also we note that $m_2 < n_2$. Hence*

$$f'' = \frac{p_2'q_2 - p_2q_2'}{q_2^2} = \frac{p_3}{q_3}, \text{ say,}$$

*where $p_3$ and $q_3$ are polynomials of degree $m_3 = m_2+n_2-1 = m_1+3n-2$ and $n_3 = 2n_2 = 4n$. Also we see that $m_3 < n_3$.*

*Let $\psi = f^2 f^{(2)} = P/Q$. Then $P$, $Q$ are polynomials of degree $2m + m_3$ and $2n + n_3$ respectively and $2m + m_3 < 2n + n_3$. Therefore $\psi$ is nonconstant.*

*Now $\psi - a = (P - aQ)/Q$ and the degree of $P - aQ$ is equal to the degree of $Q$. If $\psi - a$ has no zero then $P - aQ$ and $Q$ share $0$ CM (counting multiplicites) and so $P - aQ \equiv AQ$, where $A$ is a constant. Therefore $\psi = A - a$, which is impossible. So $\psi - a$ must have some zero.*

**Case 2.** *Let $m = n + 1$. Then*

$$f = \alpha z + \beta + \frac{p_1}{q},$$

*where $\alpha$, $\beta$ are constants and $p_1$ is a polynomial of degree $m_1 < n$.*

*Now $f'' = p_3/q_3$, where $p_3$ and $q_3$ are polynomials of degree $m_3 = m_1 + 3n - 2$ and $n_3 = 4n$ respectively and $m_3 < n_3$.*

*If $\psi = P/Q$ then $P$, $Q$ are polynomials of degree $2m + m_3$ and $2n + n_3$ respectively. We see that $2m + m_3 = 5n + m_1 < 6n = 2n + n_3$ and so $\psi$ is nonconstant. Therefore as Case 1 $\psi - a$ must have some zero.*

**Case 3.** *Let $m \neq n, n + 1$. Then*

$$f' = \frac{pq' - p'q}{q^2} = \frac{p_4}{q_4}, \text{ say,}$$

*where $p_4$, $q_4$ are polynomials of degree $m_4 = m + n - 1$ and $n_4 = 2n$. Also we note that $m_4 \neq n_4$.*

*Hence*

$$f'' = \frac{p_4'q_4 - p_4q_4'}{q_4^2} = \frac{p_5}{q_5}, \text{ say,}$$

*where $p_5$, $q_5$ are polynmials of degree $m_5 = m_4 + n_4 - 1 = m + 3n - 2$ and $n_5 = 2n_4 = 4n$.*

*If $\psi = P/Q$ then $P$, $Q$ are polynomials of degree $2m + m_5$ and $2n + n_5$ respectively. Clearly $2m + m_5 \neq 2n + n_5$ because otherwise $m = n + 2/3$, which is impossible. So $\psi$ is nonconstant. Also we see that $\psi - a = (P - aQ)/Q$, where the degree of $P - aQ$ is not less than that of $Q$. If $\psi - a$ has no zero then as per Case 1 $\psi$ becomes a constant, which is impossible. So $\psi - a$ must have some zero. This proves the lemma.*

□

**Lemma 2.8.** *Let $f$ be a meromorphic function in $\mathbb{C}$ such that $f^{(k)} \not\equiv 0$ and $a(\neq 0)$ be a finite constant. Then $f^{(k)} + af^{-n}$ must have some zero, where $k$ and $n(\geq k)$ are positive integers.*

*Proof.* First we assume that $k = 1$. Then by *Lemmas 2.5* and *2.6* we see that $f^n f' + a$ must have some zero. Since a zero of $f^n f' + a$ is not a pole or a zero of $f$, it follows that a zero of $f^n f' + a$ is a zero of $f' + af^{-n}$.

Now we assume that $k = 2$. Then by *Lemmas 2.3, 2.4* and *2.7* we see that $f^n f^{(2)} + a$ must have some zero. As the preceding paragraph a zero of $f^n f^{(2)} + a$ is a zero of $f^{(2)} + af^{-n}$.

Finally we assume that $k \geq 3$. Then by *Lemma 2.3* $f^n f^{(k)} + a$ must have some zero. Since a zero of $f^n f^{(k)} + a$ is a zero of $f^{(k)} + af^{-n}$, the lemma is proved. □

**Lemma 2.9.** *Let $f$ be a nonconstant meromorphic function in $\mathbb{C}$ such that $f$ has no zero and has no pole of multiplicity less than $3 + k$. Then $f^{(k)} - 1$ must have some simple zero, where $k$ is a positive integer.*

*Proof.* Since $N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f)$ and $m(r, f^{(k)}) \leq m(r, f) + S(r, f)$, we get

$$\begin{aligned} T(r, f^{(k)}) &\leq T(r, f) + k\overline{N}(r, f) + S(r, f) \\ &\leq T(r, f) + \frac{k}{3+k} N(r, f) + S(r, f) \\ &\leq \frac{3+2k}{3+k} T(r, f) + S(r, f). \end{aligned}$$

Since $f$ has no zero and no pole of multiplicity less than $3 + k$, we get by Milloux inequality ([8, p. 57])

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{N}(r, 1; f^{(k)}) + S(r, f) \\ &\leq \frac{1}{3+k} T(r, f) + \overline{N}(r, 1; f^{(k)}) + S(r, f). \end{aligned}$$

If possible, suppose that $f^{(k)} - 1$ has no simple zero. Then we get from above

$$\begin{aligned} T(r, f) &\leq \frac{1}{3+k} T(r, f) + \frac{1}{2} N(r, 1; f^{(k)}) + S(r, f) \\ &\leq \left\{ \frac{1}{3+k} + \frac{3+2k}{2(3+k)} \right\} T(r, f) + S(r, f) \end{aligned}$$

and so

$$\frac{1}{2(3+k)} T(r, f) \leq S(r, f),$$

a contradiction. This proves the lemma. □

**Lemma 2.10.** [4, 20] *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $\mathcal{D}$ and let the zeros of $f$ be of multiplicity not less than $k$ ( a positive integer) for each $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$ then for $0 \leq \alpha < k$ there exist a sequence of complex numbers $z_j \to z_0$, a sequence of functions $f_j \in \mathcal{F}$, and a sequence of positive numbers $\rho_j \to 0$ such that*

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

*converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Moreover the order of $g$ is not greater than two and the zeros of $g$ are of multiplicity not less than $k$.*

**Note 1.** *If each $f \in \mathcal{F}$ has no zero then $g$ also has no zero and in this case we can choose $\alpha$ to be any finite real number.*

## 3. PROOFS OF THE THEOREMS

In this section we discuss the proofs of the theorems.

*Proof of Theorem 1.1.* If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then $\mathcal{F}_1 = \{1/f : f \in \mathcal{F}\}$ is not normal at $z_0 \in \mathcal{D}$. Let $\alpha = 1$. Then by *Lemma 2.10* there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-1} f_j^{-1}(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic fucntion $g(\zeta)$ in $\mathbb{C}$. Also the order of $g$ does not exceed two and $g$ has no simple zero. Again by Hurwitz's theorem $g$ has no simple pole.

By *Lemmas 2.1* and *2.2* we see that there exists $\zeta_0 \in \mathbb{C}$ such that

$$g'(\zeta_0) + a = 0. \tag{3.1}$$

Since $\zeta_0$ is not a pole of $g$, it follows that $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighbourhood of $\zeta_0$. We also see that $\frac{-1}{g^2(\zeta)}\{g'(\zeta) + a\}$ is the uniform limit of $\rho_j^2\{f_j' - af_j^2 - b\}$ in some neighbourhood of $\zeta_0$.

In view of (3.1) and Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that $f_j'(\zeta_j) - af_j^2(\zeta_j) - b = 0$. So by the given condition

$$|g_j(\zeta_j)| = \frac{1}{\rho_j} \cdot \frac{1}{|f_j(z_j + \rho_j \zeta_j)|} \geq \frac{1}{\rho_j M}.$$

Since $\zeta_0$ is not a pole of $g$, there exists a positive number $K$ such that in some neighbourhood of $\zeta_0$ we get $|g(\zeta)| \leq K$.

Since $g_j(\zeta)$ converges uniformly to $g(\zeta)$ in some neighbourhood of $\zeta_0$, we get for all large values of $j$ and for all $\zeta$ in that neighbourhood of $\zeta_0$

$$|g_j(\zeta) - g(\zeta)| < 1.$$

Since $\zeta_j \to \zeta$, we get for all large values of $j$

$$K \geq |g(\zeta_j)| \geq |g_j(\zeta_j)| - |g(\zeta_j) - g_j(\zeta_j)| > \frac{1}{\rho_j M} - 1,$$

which is a contradiction. This proves the theorem. □

*Proof of Theorem 1.2.* Let $\alpha = k$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by *Lemma 2.10* and *Note 1* there exists a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Now by conditions (i) and (ii) and by Hurwitz's theorem we see that $g(\zeta)$ has no zero and has no pole of multiplicity less than $3 + k$.

Now by *Lemma 2.9* $g^{(k)}(\zeta) - 1$ has a simple zero at a point $\zeta_0 \in \mathbb{C}$. Since $\zeta_0$ is not a pole of $g(\zeta)$, in some neighbourhood of $\zeta_0$, $g_j(\zeta)$ converges uniformly to $g(\zeta)$.

Since

$$\begin{aligned} g_j^{(k)}(\zeta) - 1 + \sum_{i=0}^{k-1} a_i \rho_j^{k-i} g_j^{(i)}(\zeta) &= f_j^{(k)}(z_j + \rho_j \zeta) + \sum_{i=0}^{k-1} a_i f_j^{(i)}(z_j + \rho_j \zeta) - 1 \\ &= H(f_j(z_j + \rho_j \zeta)) - 1 \end{aligned}$$

and $\sum_{i=0}^{k-1} a_i \rho_j^{k-i} g_j^{(i)}(\zeta)$ converges uniformly to zero in some neighbourhood of $\zeta_0$, it follows that $g^{(k)}(\zeta) - 1$ is the uniform limit of $H(f_j(z_j + \rho_j \zeta)) - 1$.

Since $\zeta_0$ is a simple zero of $g^{(k)}(\zeta) - 1$, by Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that $\zeta_j$ is a simple zero of $H(f_j(z_j + \rho_j \zeta)) - 1$. So by the given condition $|f_j(z_j + \rho_j \zeta_j)| \geq M$ for all large values of $j$.

Hence for all large values of $j$ we get $|g_j(\zeta_j)| \geq M/\rho_j^k$ and as the last part of the proof of *Theorem 1.1* we arrive at a contradiction. This proves the theorem. □

*Proof of Theorem 1.4.* Let $\alpha = k/(1+n) < 1$. If possible suppose that $\mathcal{F}$ is not normal at $z_0 \in \mathcal{D}$. Then by *Lemma 2.10* there exist a sequence of functions $f_j \in \mathcal{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that

$$g_j(\zeta) = \rho_j^{-\alpha} f_j(z_j + \rho_j \zeta)$$

converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ in $\mathbb{C}$. Also $g$ has no zero of multiplicity less than $k$. So $g^{(k)} \not\equiv 0$ and by *Lemma 2.8* we get

$$g^{(k)}(\zeta_0) + \frac{a}{g^n(\zeta_0)} = 0 \tag{3.2}$$

for some $\zeta_0 \in \mathbb{C}$.

Clearly $\zeta_0$ is neither a zero nor a pole of $g$. So in some neighbourhood of $\zeta_0$, $g_j(\zeta)$ converges uniformly to $g(\zeta)$.

Now in some neighbourhood of $\zeta_0$ we see that $g^{(k)}(\zeta) + ag^{-n}(\zeta)$ is the uniform limit of

$$g_j^{(k)} + ag_j^{-n}(\zeta) - \rho_j^{n\alpha} b = \rho_j^{\frac{nk}{1+n}} \left\{ f_j^{(k)}(z_j + \rho_j \zeta) + af_j^{-n}(z_j + \rho_j \zeta) - b \right\}.$$

By (3.2) and Hurwitz's theorem there exists a sequence $\zeta_j \to \zeta_0$ such that for all large values of $j$

$$f_j^{(k)}(z_j + \rho_j \zeta_j) + af_j^{-n}(z_j + \rho_j \zeta_j) = b.$$

Therefore for all large values of $j$ it follows from the given condition $|g_j(\zeta_j)| \geq M/\rho_j^{\alpha}$ and as in the last part of the proof of *Theorem 1.1* we arrive at a contradiction. This proves the theorem. □

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