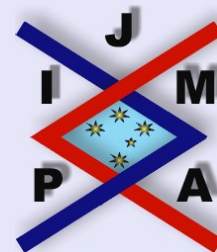


INEQUALITIES INVOLVING MULTIPLIERS FOR MULTIVALENT HARMONIC FUNCTIONS

H. ÖZLEM GÜNEY AND OM P. AHUJA

Department of Mathematics
Faculty of Science and Art
Dicle University
Diyarbakir 21280, Turkey
EMail: ozlemg@dicle.edu.tr

Kent State University
Department of Mathematical Sciences
14111, Claridon-Troy Road
Burton, Ohio 44021, U.S.A.
EMail: oahuja@kent.edu



volume 7, issue 5, article 190,
2006.

*Received 17 May, 2006;
accepted 30 August, 2006.*

Communicated by: N.E. Cho

[Abstract](#)

[Contents](#)



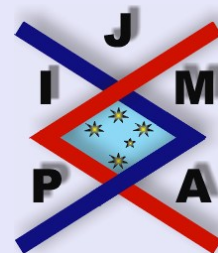
[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)





Abstract

We introduce inequalities involving multipliers for complex-valued multivalent harmonic functions, using two sequences of positive real numbers. By specializing those sequences, we determine representation theorems, distortion bounds, integral convolutions, convex combinations and neighborhoods for such functions. The theorems presented, in many cases, confirm or generalize various well-known results for corresponding classes of multivalent or univalent harmonic functions.

2000 Mathematics Subject Classification: Primary 30C45; Secondary 30C50.

Key words: Multivalent harmonic, Multivalent harmonic starlike, Multivalent harmonic convex, Multiplier, Integral convolution, Neighborhood.

The authors wish to thank the referee for suggesting certain improvements in the paper.

Contents

1	Introduction	3
2	Main Results	8
	References	

Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güneş and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 2 of 21

1. Introduction

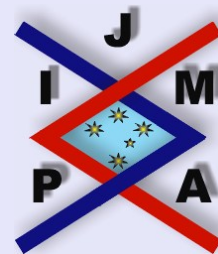
A continuous complex-valued function $f = u+iv$ defined in a simply connected complex domain \mathbb{D} is said to be harmonic in \mathbb{D} if both u and v are real harmonic in \mathbb{D} . Such functions admit the representation $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} . In [5], it was observed that $f = h + \bar{g}$ is locally univalent and sense preserving if and only if $|g'(z)| < |h'(z)|$, $z \in \mathbb{D}$.

The study of harmonic functions which are multivalent in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ was initiated by Duren, Hengartner and Laugesen [6]. However, passing from harmonic univalent functions to the harmonic multivalent functions turns out to be quite non-trivial. In view of the argument principle for harmonic functions obtained in [6], the second author and Jahangiri [1, 2] introduced and studied certain subclasses of the family $H(m)$, $m \geq 1$, of all m -valent harmonic and orientation preserving functions in \mathbb{U} . A function f in $H(m)$ can be expressed as $f = h + \bar{g}$, where h and g are analytic functions of the form

$$(1.1) \quad h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1},$$
$$g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1.$$

The class $H(1)$ of harmonic univalent functions was studied by Clunie and Sheil-Small [5].

Let $SH(m, \alpha)$, $m \geq 1$ and $0 \leq \alpha < 1$ denote the class of functions $f =$



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 3 of 21

$h + \bar{g} \in H(m)$ which satisfy the condition

$$(1.2) \quad \frac{\partial}{\partial \theta} (\arg(f(re^{i\theta}))) \geq m\alpha,$$

for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$. A function f in $SH(m, \alpha)$ is called an m -valent harmonic starlike function of order α . Also, let $TH(m, \alpha)$, $m \geq 1$, denote the class of functions $f = h + \bar{g} \in SH(m, \alpha)$ so that h and g are of the form

$$(1.3) \quad h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1},$$

$$g(z) = \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \quad |b_m| < 1.$$

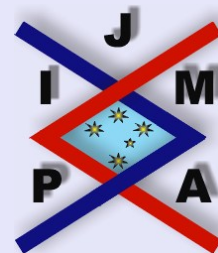
The class $TH(m, \alpha)$ was studied by second author and Jahangiri in [1, 2]. In particular, they stated the following:

Theorem A. *Let $f = h + \bar{g}$ be given by (1.3). Then f is in $TH(m, \alpha)$ if and only if*

$$(1.4) \quad \sum_{n=1}^{\infty} \left[\frac{n-1+m(1-\alpha)}{m(1-\alpha)} |a_{n+m-1}| + \frac{n-1+m(1+\alpha)}{m(1-\alpha)} |b_{n+m-1}| \right] \leq 2,$$

where $a_m = 1$ and $m \geq 1$.

Analogous to $TH(m, \alpha)$ is the class $KH(m, \alpha)$ of m -valent harmonic convex functions of order α , $0 \leq \alpha < 1$. More precisely, a function $f = h + \bar{g}$,



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 4 of 21

where h and g are of the form (1.3), is in $KH(m, \alpha)$ if and only if it satisfies the condition

$$\frac{\partial}{\partial \theta} \left(\arg \left(\frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) \geq m\alpha,$$

for each $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, and $0 \leq r < 1$.

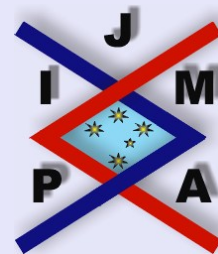
Theorem B ([4]). Let $f = h + \bar{g}$ be given by (1.3). Then f is in $KH(m, \alpha)$ if and only if

$$(1.5) \quad \sum_{n=1}^{\infty} \frac{n+m-1}{m^2(1-\alpha)} [(n-1+m(1-\alpha))|a_{n+m-1}| \\ + (n-1+m(1+\alpha))|b_{n+m-1}|] \leq 2,$$

where $a_m = 1$ and $m \geq 1$.

Inequalities (1.4) and (1.5) as well as several such known inequalities in the literature are the motivating forces for introducing a multiplier family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ for $m \geq 1$. A function $f = h + \bar{g}$, where h and g are given by (1.3), is said to be in the multiplier family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ if there exist sequences $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ of positive real numbers such that

$$(1.6) \quad \sum_{n=1}^{\infty} \left[\frac{c_{n+m-1}}{m} |a_{n+m-1}| + \frac{d_{n+m-1}}{m} |b_{n+m-1}| \right] \leq 2, \\ c_m = m, d_m |b_m| < m.$$



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 5 of 21

The multipliers $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ provide a transition from multivalent harmonic starlike functions to multivalent harmonic convex functions, including many more subclasses of $H(m)$ and $H(1)$. For example,

$$(1.7) \quad F_m \left(\left\{ \frac{n-1+m(1-\alpha)}{1-\alpha} \right\}, \left\{ \frac{n-1+m(1+\alpha)}{1-\alpha} \right\} \right) \equiv TH(m, \alpha),$$

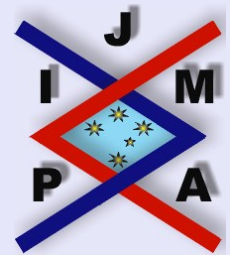
$$(1.8) \quad F_m \left(\left\{ \frac{(n+m-1)(n-1+m(1-\alpha))}{m(1-\alpha)} \right\}, \left\{ \frac{(n+m-1)(n-1+m(1+\alpha))}{m(1-\alpha)} \right\} \right) \equiv KH(m, \alpha),$$

$$(1.9) \quad F_m(\{n+m-1\}, \{n+m-1\}) \equiv TH(m, 0) := TH(m),$$

$$(1.10) \quad F_m \left(\left\{ \frac{(n+m-1)^2}{m} \right\}, \left\{ \frac{(n+m-1)^2}{m} \right\} \right) \equiv KH(m, 0) \\ := KH(m),$$

$$(1.11) \quad F_1(\{n\}, \{n\}) \equiv TH(1, 0) = TH,$$

$$(1.12) \quad F_1(\{n^2\}, \{n^2\}) \equiv KH(1, 0) := KH,$$



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

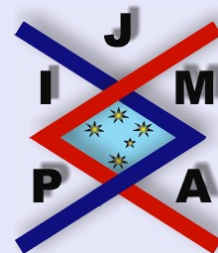
Page 6 of 21

$$(1.13) \quad F_1(\{n^p\}, \{n^p\}) := F(\{n^p\}, \{n^p\}), \quad p > 0,$$

$$(1.14) \quad F_1(\{c_n\}, \{d_n\}) := F(\{c_n\}, \{d_n\}).$$

While (1.7), (1.9) and (1.11) follow immediately from Theorem A, (1.8), (1.10) and (1.12) are consequences of Theorem B. Note that TH and KH in (1.11) and (1.12) were studied in [9] as well as [10]. Also, by letting $m = 1$, $\alpha = 0$, $c_n = d_n = n^p$ for $p > 0$ and $b_1 = 0$, the classes $F_1(\{n^p\}, \{n^p\})$ were studied in [8]. Finally, (1.14) follows from (1.6) by setting $m = 1$ which was studied in [3].

In this paper, we determine representation theorems, distortion bounds, convolutions, convex combinations and neighborhoods of functions in $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$. As illustrations of our results, the corresponding results for certain families are presented in the corollaries.



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 7 of 21

2. Main Results

If $(n + m - 1) \leq c_{n+m-1}$ and $(n + m - 1) \leq d_{n+m-1}$, then by Theorem A we have

$$F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\}) \subset TH(m).$$

Consequently, the functions $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ are sense-preserving, harmonic and multivalent in \mathbb{U} . We first observe that if

$$f_1(z) = z^m - \sum_{n=2}^{\infty} |a_{1(n+m-1)}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{1(n+m-1)}| \bar{z}^{n+m-1}$$

and

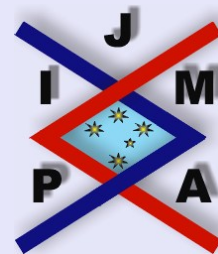
$$f_2(z) = z^m - \sum_{n=2}^{\infty} |a_{2(n+m-1)}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{2(n+m-1)}| \bar{z}^{n+m-1}$$

are in $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ and $0 \leq \lambda \leq 1$, then so is the linear combination $\lambda f_1 + (1 - \lambda) f_2$ by (1.6). Therefore, $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ is a convex family.

Next we determine the extreme points of the closed convex hull of the family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, denoted by $clcoF_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.1. *A function $f = h + \bar{g}$ is in $clcoF_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ if and only if f has the representation*

$$(2.1) \quad f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z)),$$



Inequalities Involving
Multipliers For Multivalent
Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 8 of 21

where

$$(n + m - 1) \leq c_{n+m-1}, \quad (n + m - 1) \leq d_{n+m-1}, \quad \lambda_{n+m-1} \geq 0,$$

$$\mu_{n+m-1} \geq 0, \quad \sum_{n=1}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}) = 1, \quad h_m(z) = z^m,$$

$$h_{n+m-1}(z) = z^m - \frac{m}{c_{n+m-1}} z^{n+m-1} \quad \text{and} \quad g_{n+m-1}(z) = z^m + \frac{m}{d_{n+m-1}} \bar{z}^{n+m-1}.$$

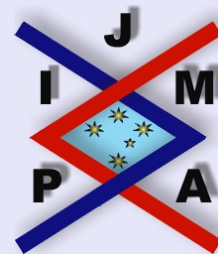
In particular, the extreme points of $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ are $\{h_{n+m-1}\}, \{g_{n+m-1}\}$.

Proof. For functions f of the form (2.1) we have

$$\begin{aligned} f(z) &= \lambda_m h_m(z) + \sum_{n=2}^{\infty} \lambda_{n+m-1} \left(z^m - \frac{m}{c_{n+m-1}} z^{n+m-1} \right) \\ &\quad + \sum_{n=1}^{\infty} \mu_{n+m-1} \left(z^m + \frac{m}{d_{n+m-1}} \bar{z}^{n+m-1} \right) \\ &= z^m - \sum_{n=2}^{\infty} \lambda_{n+m-1} \frac{m}{c_{n+m-1}} z^{n+m-1} + \sum_{n=1}^{\infty} \mu_{n+m-1} \frac{m}{d_{n+m-1}} \bar{z}^{n+m-1}. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} \lambda_{n+m-1} \frac{m}{c_{n+m-1}} c_{n+m-1} + \sum_{n=1}^{\infty} \mu_{n+m-1} \frac{m}{d_{n+m-1}} d_{n+m-1} \\ = \sum_{n=2}^{\infty} m \lambda_{n+m-1} + \sum_{n=1}^{\infty} m \mu_{n+m-1} \leq m, \end{aligned}$$



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 9 of 21

and so $f \in clcoF_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Conversely, suppose $f \in clcoF_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$. We set

$$\lambda_{n+m-1} = \frac{c_{n+m-1}}{m} |a_{n+m-1}|, \quad (n = 2, 3, \dots),$$

$$\mu_{n+m-1} = \frac{d_{n+m-1}}{m} |b_{n+m-1}|, \quad (n = 1, 2, 3, \dots),$$

and

$$\lambda_m = 1 - \sum_{n=2}^{\infty} \lambda_{n+m-1} - \sum_{n=1}^{\infty} \mu_{n+m-1}.$$

Therefore, by using routine computations, f can be written as

$$f(z) = \sum_{n=1}^{\infty} (\lambda_{n+m-1} h_{n+m-1}(z) + \mu_{n+m-1} g_{n+m-1}(z)).$$

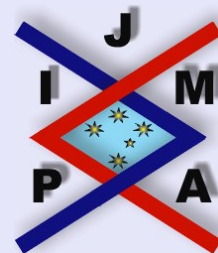
□

In view of (1.7), Theorem 2.1 yields:

Corollary 2.2 ([2]). *A function $f = h + \bar{g}$ is in $clcoTH(m, \alpha)$ if and only if f can be expressed in the form (2.1), where*

$$h_m(z) = z^m, \quad h_{n+m-1}(z) = z^m - \frac{m(1-\alpha)}{n-1+m(1-\alpha)} z^{n+m-1}, \quad (n = 2, 3, \dots),$$

$$g_{n+m-1}(z) = z^m + \frac{m(1-\alpha)}{n-1+m(1+\alpha)} \bar{z}^{n+m-1}, \quad (n = 1, 2, 3, \dots)$$



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 10 of 21

and

$$\sum_{n=1}^{\infty} (\lambda_{n+m-1} + \mu_{n+m-1}) = 1, \quad \lambda_{n+m-1} \geq 0, \quad \mu_{n+m-1} \geq 0.$$

Our next result provides distortion bounds for the functions in $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.3. *Let $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ be increasing sequences of positive real numbers so that*

$$c_{m+1} \leq d_{m+1}, \quad (n + m - 1) \leq c_{n+m-1} \quad \text{and} \quad (n + m - 1) \leq d_{n+m-1}$$

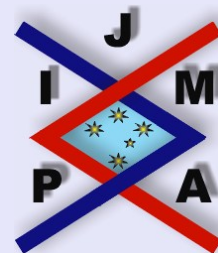
for all $n \geq 2$. If $f \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, then

$$(1 - |b_m|)r^m - \left(\frac{m - d_m|b_m|}{c_{m+1}} \right) r^{m+1} \leq |f(z)| \leq (1 + |b_m|)r^m + \left(\frac{m - d_m|b_m|}{c_{m+1}} \right) r^{m+1}.$$

The bounds given above are sharp for the functions

$$f(z) = z^m \pm |b_m| \bar{z}^m + \left(\frac{m - d_m|b_m|}{c_{m+1}} \right) \bar{z}^{m+1}, \quad d_m|b_m| < 1.$$

Proof. Let $f \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$. Taking the absolute value of f , we



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 11 of 21

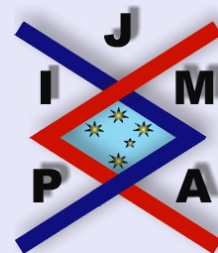
obtain

$$\begin{aligned}
 |f(z)| &= \left| z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \bar{z}^{n+m-1} \right| \\
 &\leq r^m + \sum_{n=2}^{\infty} |a_{n+m-1}| r^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| r^{n+m-1} \\
 &= (1 + |b_m|) r^m + \sum_{n=2}^{\infty} (|a_{n+m-1}| + |b_{n+m-1}|) r^{n+m-1} \\
 &\leq (1 + |b_m|) r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} c_{m+1} (|a_{n+m-1}| + |b_{n+m-1}|) r^{m+1} \\
 &\leq (1 + |b_m|) r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} (c_{m+1} |a_{n+m-1}| + d_{m+1} |b_{n+m-1}|) r^{m+1} \\
 &\leq (1 + |b_m|) r^m + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} (c_{n+m-1} |a_{n+m-1}| + d_{n+m-1} |b_{n+m-1}|) r^{m+1} \\
 &\leq (1 + |b_m|) r^m + \frac{1}{c_{m+1}} (m - d_m |b_m|) r^{m+1}.
 \end{aligned}$$

We omit the proof of the left side of the inequality as it is similar to that of the right side. \square

Corollary 2.4. *If $f \in TH(m, \alpha)$, then*

$$|f(z)| \leq (1 + |b_m|) r^m + \frac{m(1 - \alpha - (1 + \alpha)|b_m|)}{1 + m(1 - \alpha)} r^{m+1},$$



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 12 of 21

and

$$|f(z)| \geq (1 - |b_m|)r^m - \frac{m(1 - \alpha - (1 + \alpha)|b_m|)}{1 + m(1 - \alpha)}r^{m+1},$$

where $|z| = r < 1$.

The following covering result follows from the left hand inequality in Theorem 2.3.

Corollary 2.5. *Let f be as in Theorem 2.3. Then*

$$\left\{ w : |w| < \frac{1}{c_{m+1}}(c_{m+1} - m - (c_{m+1} - d_m)|b_m|) \right\} \subset f(\mathbb{U}).$$

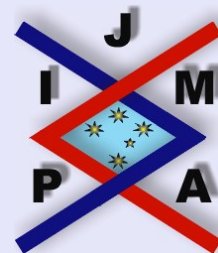
Corollary 2.6. *If $f \in TH(m, \alpha)$, then*

$$\left\{ w : |w| < \frac{1 + (2m\alpha - 1)|b_m|}{1 + m(1 - \alpha)} \right\} \subset f(\mathbb{U}).$$

Remark 1. *For $\alpha = 0$, the corresponding results in Corollary 2.4 and Corollary 2.6 were also found in [1].*

In the next result, we find the convex combinations of the members of the family $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.7. *If $(n + m - 1) \leq c_{n+m-1}$ and $(n + m - 1) \leq d_{n+m-1}$ for all $n + m - 1 \geq 2$, then $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ is closed under convex combinations.*



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 13 of 21

Proof. Consider

$$f_i(z) = z^m - \sum_{n=2}^{\infty} |a_{i_{n+m-1}}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{i_{n+m-1}}| \bar{z}^{n+m-1}$$

for $i = 1, 2, \dots$. If $f_i \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ then

$$(2.2) \quad \sum_{n=2}^{\infty} c_{n+m-1} |a_{i_{n+m-1}}| + \sum_{n=1}^{\infty} d_{n+m-1} |b_{i_{n+m-1}}| \leq m, \quad i = 1, 2, \dots$$

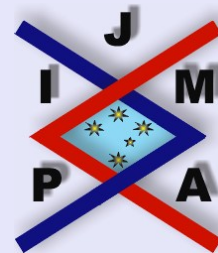
For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, we have

$$\begin{aligned} & \sum_{i=1}^{\infty} t_i f_i(z) \\ &= z^m - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_{n+m-1}}| \right) z^{n+m-1} + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_{n+m-1}}| \right) \bar{z}^{n+m-1}. \end{aligned}$$

In view of the above equality and (2.2), we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} c_{n+m-1} \left| \sum_{i=1}^{\infty} t_i |a_{i_{n+m-1}}| \right| + \sum_{n=1}^{\infty} d_{n+m-1} \left| \sum_{i=1}^{\infty} t_i |b_{i_{n+m-1}}| \right| \\ &= \sum_{i=1}^{\infty} t_i \left\{ \sum_{n=2}^{\infty} c_{n+m-1} |a_{i_{n+m-1}}| + \sum_{n=1}^{\infty} d_{n+m-1} |b_{i_{n+m-1}}| \right\} \\ &\leq \sum_{i=1}^{\infty} t_i m = m. \end{aligned}$$

Hence $\sum_{i=1}^{\infty} t_i f_i \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, by an application of (1.6). \square



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 14 of 21

In view of relations (1.7) to (1.10), we have the following results:

Corollary 2.8. *The family $TH(m, \alpha)$, $KH(m, \alpha)$, $TH(m)$ and $KH(m)$ are closed under convex combinations.*

For harmonic functions

$$(2.3) \quad f(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |b_{n+m-1}| \bar{z}^{n+m-1}$$

and

$$(2.4) \quad F(z) = z^m - \sum_{n=2}^{\infty} |A_{n+m-1}| z^{n+m-1} + \sum_{n=1}^{\infty} |B_{n+m-1}| \bar{z}^{n+m-1}$$

define the integral convolution of f and F as

$$(2.5) \quad (f \diamond F)(z) = z^m - \sum_{n=2}^{\infty} \frac{|a_{n+m-1} A_{n+m-1}|}{n+m-1} z^{n+m-1} + \sum_{n=1}^{\infty} \frac{|b_{n+m-1} B_{n+m-1}|}{n+m-1} \bar{z}^{n+m-1}.$$

In the following result, we show the integral convolution property of the class $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$.

Theorem 2.9. *Let $(n+m-1) \leq c_{n+m-1}$ and $(n+m-1) \leq d_{n+m-1}$ for all $n+m-1 \geq 2$. If f and F are in $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, then so is $f \diamond F$.*



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 15 of 21

Proof. Since $F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\}) \subset TH(m)$ and $F \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$, it follows that $|A_{n+m-1}| \leq 1$ and $|B_{n+m-1}| \leq 1$. Then $f \diamond F \in F_m(\{c_{n+m-1}\}, \{d_{n+m-1}\})$ because

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m(n+m-1)} |a_{n+m-1} A_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m(n+m-1)} |b_{n+m-1} B_{n+m-1}| \\ & \leq \sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m(n+m-1)} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m(n+m-1)} |b_{n+m-1}| \\ & \leq \sum_{n=2}^{\infty} \frac{c_{n+m-1}}{m} |a_{n+m-1}| + \sum_{n=1}^{\infty} \frac{d_{n+m-1}}{m} |b_{n+m-1}| \leq 2. \end{aligned}$$

□

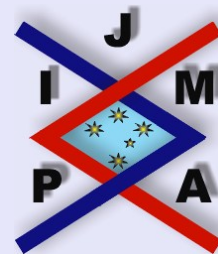
Corollary 2.10. *If f and F are in $TH(m, \alpha)$, $KH(m, \alpha)$, $TH(m)$ and $KH(m)$, then so is $f \diamond F$.*

The δ -neighborhood of the functions $f = h + \bar{g}$ in $F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\})$ is defined as the set $N_\delta(f)$ consisting of functions

$$F(z) = z^m + B_m \bar{z}^m + \sum_{n=2}^{\infty} (A_{n+m-1} z^{n+m-1} + B_{n+m-1} \bar{z}^{n+m-1})$$

such that

$$\begin{aligned} \sum_{n=2}^{\infty} [(n+m-1)(|a_{n+m-1} - A_{n+m-1}| + |b_{n+m-1} - B_{n+m-1}|)] \\ + m|b_m - B_m| \leq \delta, \quad \delta > 0. \end{aligned}$$



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 16 of 21

Our next result guarantees that the functions in a neighborhood of

$$F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\})$$

are multivalent harmonic starlike functions.

Theorem 2.11. *Let $\{c_{n+m-1}\}$ and $\{d_{n+m-1}\}$ be increasing sequences of real numbers so that $c_{m+1} \leq d_{m+1}$, $(n+m-1) \leq c_{n+m-1}$ and $(n+m-1) \leq d_{n+m-1}$ for all $n \geq 2$. If*

$$\delta = \frac{m}{c_{m+1}}(c_{m+1} - 1 - (c_{m+1} - d_m)|b_m|),$$

then

$$N_\delta(F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\})) \subset TH(m).$$

Proof. Suppose

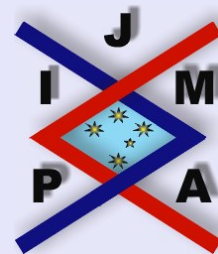
$$f = h + \bar{g} \in F_m(\{(n+m-1)c_{n+m-1}\}, \{(n+m-1)d_{n+m-1}\}).$$

Let $F = H + \bar{G} \in N_\delta(f)$ where

$$H(z) = z^m + \sum_{n=2}^{\infty} A_{n+m-1} z^{n+m-1} \quad \text{and} \quad G(z) = \sum_{n=1}^{\infty} B_{n+m-1} z^{n+m-1}.$$

We need to show that $F \in TH(m)$. It suffices to show that F satisfies the condition

$$M(F) := \sum_{n=2}^{\infty} (n+m-1)(|A_{n+m-1}| + |B_{n+m-1}|) + m|B_m| \leq m$$



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 17 of 21

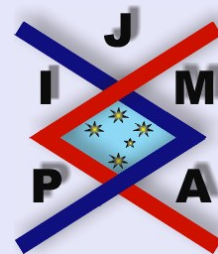
Note that

$$\begin{aligned}
 M(F) &\leq \sum_{n=2}^{\infty} (n+m-1) [|A_{n+m-1} - a_{n+m-1}| + |B_{n+m-1} - b_{n+m-1}|] \\
 &\quad + m|B_m - b_m| + \sum_{n=2}^{\infty} (n+m-1)(|a_{n+m-1}| + |b_{n+m-1}|) + m|b_m| \\
 &\leq \delta + m|b_m| + \sum_{n=2}^{\infty} (n+m-1)(|a_{n+m-1}| + |b_{n+m-1}|) \\
 &= \delta + m|b_m| + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} (c_{m+1}(n+m-1)|a_{n+m-1}| \\
 &\quad + c_{m+1}(n+m-1)|b_{n+m-1}|) \\
 &\leq \delta + m|b_m| + \frac{1}{c_{m+1}} \sum_{n=2}^{\infty} ((n+m-1)c_{n+m-1}|a_{n+m-1}| \\
 &\quad + (n+m-1)d_{n+m-1}|b_{n+m-1}|) \\
 &\leq \delta + m|b_m| + \frac{1}{c_{m+1}} (m(1-d_m|b_m|)).
 \end{aligned}$$

But, the last expression is never greater than m provided that

$$\begin{aligned}
 \delta &\leq m - m|b_m| - \frac{1}{c_{m+1}} (m(1-d_m|b_m|)) \\
 &= \frac{m}{c_{m+1}} (c_{m+1} - 1 - (c_{m+1} - d_m)|b_m|).
 \end{aligned}$$

□



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 18 of 21

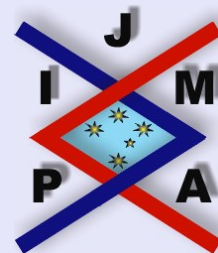
Corollary 2.12. *If*

$$\delta = \frac{m - (m - 2m^2\alpha)|b_m|}{1 + m(1 - \alpha)},$$

then $N_\delta(KH(m, \alpha)) \subset TH(m)$.

Letting $\alpha = 0$ and $m = 1$, Corollary 2.12 yields the following interesting result.

Corollary 2.13. $N_{\frac{1}{2}(1-|b_1|)}(KH) \subset TH$.



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 19 of 21

References

- [1] O.P. AHUJA AND J.M. JAHANGIRI, Multivalent harmonic starlike functions, *Ann.Univ. Mariae Curie-Sklodowska*, Section A, **55**(1) (2001), 1–13.
- [2] O.P. AHUJA AND J.M. JAHANGIRI, Errata to "Multivalent harmonic starlike functions" [*Ann.Univ. Mariae Curie-Sklodowska*, Vol. LV, 1 Sectio A 55(2001), 1-13], *Ann.Univ. Mariae Curie-Sklodowska*, Sectio A, **56**(1) (2002), 105.
- [3] O.P. AHUJA AND J.M. JAHANGIRI, Certain multipliers of univalent harmonic functions, *Appl. Math. Letters*, **18** (2005), 1319–1324.
- [4] O.P. AHUJA AND J.M. JAHANGIRI, Multivalent harmonic convex functions, *submitted for publication*.
- [5] J. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, **9** (1984), 3–25.
- [6] P.L. DUREN, W. HENGARTNER AND R.S. LAUGESSEN, The argument principles harmonic functions, *Amer. Math. Monthly*, (5) **103** (1996), 411–425.
- [7] A. GABRIELA AND A. LAZINSKA, On some generalization of coefficient conditions for complex harmonic mappings, *Demonstratio Math.*, **37**(2) (2004), 317–326.
- [8] A. GANCZAR, On harmonic univalent mappings with small coefficients, *Demonstratio Math.*, **34**(3) (2001), 549–558.



Inequalities Involving Multipliers For Multivalent Harmonic Functions

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



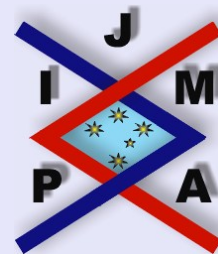
Go Back

Close

Quit

Page 20 of 21

- [9] J.M. JAHANGIRI, Harmonic functions starlike in the unit disc, *J. Math. Anal. Appl.*, **235** (1999), 470–477.
- [10] H. SILVERMAN AND E.M. SILVIA, Subclasses of harmonic univalent functions, *New Zealand J. Math.*, **28** (1999), 275–284.



**Inequalities Involving
Multipliers For Multivalent
Harmonic Functions**

H. Özlem Güney and Om P. Ahuja

Title Page

Contents



Go Back

Close

Quit

Page 21 of 21