



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 1 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

# INEQUALITIES FOR THE MAXIMUM MODULUS OF THE DERIVATIVE OF A POLYNOMIAL

A. AZIZ AND B.A. ZARGAR

Department of Mathematics  
University of Kashmir, Srinagar-India  
EMail: zargarba3@yahoo.co.in

*Received:* 17 May, 2006

*Accepted:* 18 December, 2006

*Communicated by:* Q.I. Rahman

*2000 AMS Sub. Class.:* 30C10, 30C15.

*Key words:* Polynomial, Derivative, Bernstein Inequality, Maximum Modulus.

*Abstract:* Let  $P(z)$  be a polynomial of degree  $n$  and  $M(P, t) = \max_{|z|=t} |P(z)|$ . In this paper we shall estimate  $M(P', \rho)$  in terms of  $M(P, r)$  where  $P(z)$  does not vanish in the disk  $|z| \leq K$ ,  $K \geq 1$ ,  $0 \leq r < \rho < K$  and obtain an interesting refinement of some result of Dewan and Malik. We shall also obtain an interesting generalization as well as a refinement of well-known result of P. Turan for polynomials not vanishing outside the unit disk.

# Contents

1	Introduction and Statement of Results	3
2	Lemmas	6
3	Proof of the Theorems	10



---

**Inequalities for the  
Maximum Modulus**

A. Aziz and B.A. Zargar

vol. 8, iss. 2, art. 37, 2007

---

Title Page

Contents



Page 2 of 15

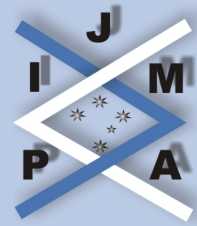
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)



Page 3 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

## 1. Introduction and Statement of Results

Let  $P(z)$  be a polynomial of degree  $n$  and let  $M(P, r) = \text{Max}_{|z|=r} |P(z)|$  and  $m(P, t) = \text{min}_{|z|=t} |P(z)|$  concerning the estimate of  $\text{max} |P'(z)|$  in terms of the  $\text{max} |P(z)|$  on the unit circle  $|z| = 1$ , we have

$$(1.1) \quad \text{Max}_{|z|=1} |P'(z)| \leq n \text{Max}_{|z|=1} |P(z)|.$$

Inequality (1.1) is a famous result known as Bernstein's Inequality (for reference see [4], [5], [10], [11]). Equality in (1.1) holds if and only if  $P(z)$  has all its zeros at the origin. So it is natural to seek improvements under appropriate assumptions on the zeros of  $P(z)$ .

If  $P(z)$  does not vanish in  $|z| < 1$ , then the inequality (1.1) can be replaced by

$$(1.2) \quad \text{Max}_{|z|=1} |P'(z)| \leq \frac{n}{2} \text{Max}_{|z|=1} |P(z)|$$

Inequality (1.2) was conjectured by Erdos and later proved by Lax [8]. On the other hand, it was shown by Turan [12] that if all the zeros of  $P(z)$  lie in  $|z| < 1$ , then

$$(1.3) \quad \text{Max}_{|z|=1} |P'(z)| \geq \frac{n}{2} \text{Max}_{|z|=1} |P(z)|.$$

As an extension of (1.2), Malik [9] showed that if  $P(z)$  does not vanish in  $|z| < K$ ,  $K \geq 1$ , then

$$(1.4) \quad \text{Max}_{|z|=1} |P'(z)| \leq \frac{n}{1+K} \text{Max}_{|z|=1} |P(z)|$$

Recently Dewan and Abdullah [6] have obtained the following generalization of inequality (1.4).



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 15

Go Back

Full Screen

Close

**Theorem A.** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having no zeros in  $|z| < K$ ,  $K \geq 1$ , then for  $0 \leq r < \rho \leq K$ ,

$$(1.5) \quad \text{Max}_{|z|=\rho} |P'(z)| \leq \frac{n(\rho + K)^{n-1}}{(K + r)^n} \left\{ 1 - \frac{K(K - \rho)(n|a_0| - K|a_1|)n}{(K^2 - \rho^2)n|a_0| + 2K^2\rho|a_1|} \right. \\ \left. \times \left( \frac{\rho - r}{K + \rho} \right) \left( \frac{K + r}{K + \rho} \right)^{n-1} \right\} \text{Max}_{|z|=r} |P(z)|$$

Inequality (1.3) was generalized by Aziz and Shah [2] by proving the following interesting result.

**Theorem B.** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having all its zeros in the disk  $|z| \leq K \leq 1$  with  $s$ -fold zeros at origin, then for  $|z| = 1$ ,

$$(1.6) \quad \text{Max}_{|z|=1} |P'(z)| \geq \frac{n + Ks}{1 + K} \text{Max}_{|z|=1} |P(z)|.$$

The result is sharp and the extremal polynomial is

$$P(z) = z^s(z + K)^{n-s}, \quad 0 < s \leq n.$$

Here in this paper, we shall first obtain the following interesting improvement of Theorem A which is also a generalization of inequality (1.4).

**Theorem 1.1.** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n > 1$ , having no zeros in  $|z| < K$ ,  $K \geq 1$ , then for  $0 \leq r \leq \rho \leq K$ ,

$$(1.7) \quad M(P', \rho) \leq \frac{n(\rho + K)^{n-1}}{(K + r)^n} \\ \times \left[ 1 - \frac{K(K - \rho)(n|a_0| - K|a_1|)n}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \left( \frac{\rho - r}{K + \rho} \right) \left( \frac{K + r}{K + \rho} \right)^{n-1} \right] M(P, r)$$

$$-n \left( \frac{r+K}{\rho+K} \right) \left[ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right] \\ \times \left\{ \left( \left( \frac{\rho+K}{r+K} \right)^n - 1 \right) - n(\rho-r) \right\} m(P, K).$$

The result is best possible and equality holds for the polynomial

$$P(z) = (z + K)^n.$$

Next we prove the following result which is a refinement of Theorem B.

**Theorem 1.2.** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having all its zeros in the disk  $|z| \leq K$ ,  $K \leq 1$  with  $t$ -fold zeros at the origin, then,

$$(1.8) \quad M(P', 1) \geq \frac{n+Kt}{1+K} M(P, 1) + \frac{n-t}{(1+K)K^t} m(P, K).$$

The result is sharp and equality holds for the polynomial

$$P(z) = z^t (z + K)^{n-t}, \quad 0 < t \leq n.$$

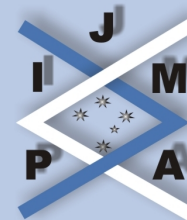
The following result immediately follows by taking  $K = 1$  in Theorem 1.2.

**Corollary 1.3.** If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having all its zeros in  $|z| \leq 1$ , with  $t$ -fold zeros at the origin, then for  $|z| = 1$ ,

$$(1.9) \quad M(P', 1) \geq \frac{n+t}{2} M(P, 1) + \frac{n-t}{2} m(P, 1).$$

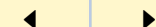
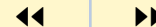
The result is best possible and equality holds for the polynomial  $P(z) = (z + K)^n$ .

*Remark 1.* For  $t = 0$ , Corollary 1.3 reduces to a result due to Aziz and Dawood [1].



Title Page

Contents

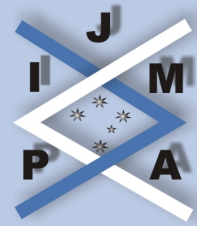


Page 5 of 15

Go Back

Full Screen

Close



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 6 of 15

[Go Back](#)

[Full Screen](#)

[Close](#)

## 2. Lemmas

For the proofs of these theorems, we require the following lemmas. The first result is due to Govil, Rahman and Schmeisser [7].

**Lemma 2.1.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having all its zeros in  $|z| \geq K \geq 1$ , then*

$$(2.1) \quad \text{Max}_{|z|=1} |P'(z)| \leq n \frac{(n|a_0| + K^2|a_1|)}{(1 + K^2)n|a_0| + 2K^2|a_1|} \text{Max}_{|z|=1} |P(z)|.$$

**Lemma 2.2.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  which does not vanish in  $|z| < K$  where  $K > 0$ , then for  $0 \leq rR \leq K^2$  and  $r \leq R$ , we have*

$$(2.2) \quad \text{Max}_{|z|=r} |P(z)| \geq \left( \frac{r + K}{R + K} \right)^n \text{Max}_{|z|=R} |P(z)| + \left[ 1 - \left( \frac{r + K}{R + K} \right)^n \right] \text{Min}_{|z|=K} |P(z)|.$$

Here the result is best possible and equality in (2.2) holds for the polynomial  $P(z) = (z + K)^n$ .

Lemma 2.2 is due to Aziz and Zargar [3].

**Lemma 2.3.** *If  $P(z) = \sum_{j=0}^n a_j z^j$  is a polynomial of degree  $n$  having no zeros in  $|z| < K$ ,  $K \geq 1$ , then for  $0 \leq r \leq \rho \leq K$ ,*

$$(2.3) \quad M(P, \rho) \leq \left( \frac{K + \rho}{K + r} \right)^n \left[ 1 - \frac{K(K - \rho)(n|a_0| - K|a_1|)n}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \left( \frac{\rho - r}{K + \rho} \right) \left( \frac{K + r}{K + \rho} \right)^{n-1} \right] M(P, r)$$



Title Page

Contents



Page 7 of 15

Go Back

Full Screen

Close

$$- \left[ \frac{(n|a_0|\rho + K^2|a_1|)(r + K)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \left\{ \left( \left( \frac{\rho + K}{r + K} \right)^n - 1 \right) - n(\rho - r) \right\} \right] m(P, K).$$

The result is best possible with equality for the polynomial  $P(z) = (z + K)^n$ .

*Proof of Lemma 2.3.* Since  $P(z)$  has no zeros in  $|z| < K$ ,  $K \geq 1$ , therefore the polynomial  $T(z) = P(tz)$  has no zeros in  $|z| < \frac{K}{t}$ , where  $0 \leq t \leq K$ . Using Lemma 2.1 for the polynomial  $T(z)$ , with  $K$  replaced by  $\frac{K}{t} \geq 1$ , we get

$$\text{Max}_{|z|=1} |T'(z)| \leq n \left\{ \frac{(n|a_0| + \frac{K^2}{t^2}|ta_1|)}{(1 + \frac{K^2}{t^2})n|a_0| + 2\frac{K^2}{t^2}|ta_1|} \right\} \text{Max}_{|z|=1} |T(z)|,$$

which implies

$$(2.4) \quad \text{Max}_{|z|=t} |P'(z)| \leq n \left\{ \frac{(n|a_0|t + K^2|a_1|)}{(t^2 + K^2)n|a_0| + 2K^2t|a_1|} \right\} \text{Max}_{|z|=t} |P(z)|.$$

Now for  $0 \leq r \leq \rho \leq K$  and  $0 \leq \theta < 2\pi$ , by (2.4) we have

$$(2.5) \quad |P(\rho e^{i\theta}) - P(re^{i\theta})| \leq \int_r^\rho |P'(te^{i\theta})| dt \\ \leq \int_r^\rho n \left\{ \frac{(n|a_0|t + K^2|a_1|)}{(t^2 + K^2)n|a_0| + 2K^2t|a_1|} \right\} \text{Max}_{|z|=t} |P(z)|.$$

Using Lemma 2.2 with  $R = t$  and noting that  $0 \leq r \leq t \leq \rho \leq K$  and  $0 \leq rt \leq K^2$ , it follows that

$$|P(\rho e^{i\theta}) - P(re^{i\theta})| \leq \int_r^\rho n \left\{ \frac{(n|a_0|t + K^2|a_1|)}{(t^2 + K^2)n|a_0| + 2K^2t|a_1|} \right\},$$



Title Page

Contents



Page 8 of 15

Go Back

Full Screen

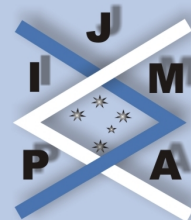
Close

$$\begin{aligned} & \left(\frac{t+K}{r+K}\right)^n \left\{ M(P, r) - \left(1 - \left(\frac{r+K}{t+K}\right)^n\right) m(P, K) \right\} dt \\ & \leq n \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \\ & \quad \times \int_r^\rho \left(\frac{t+K}{r+K}\right)^n \left\{ M(P, r) - \left(1 - \left(\frac{r+K}{t+K}\right)^n\right) m(P, K) \right\} dt. \end{aligned}$$

This gives for  $0 \leq r \leq \rho \leq K$ ,

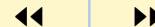
$$\begin{aligned} & M(P, \rho) \\ & \leq \left[ 1 + \frac{n(K+\rho)}{(K+r)^n} \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \int_r^\rho (K+t)^{n-1} dt \right] M(P, r) \\ & \quad - n \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \int_r^\rho \left( \left(\frac{t+K}{r+K}\right)^n - 1 \right) dt m(P, k) \\ & \leq \left[ 1 - \left\{ \frac{(K+\rho)(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \right. \\ & \quad \left. + \left\{ \frac{(K+\rho)(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \left(\frac{K+\rho}{K+r}\right)^n \right] M(P, r) \\ & \quad - n \left[ \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \int_r^\rho \left( \frac{(t+K)^{n-1}}{(r+K)^{n-1}} - 1 \right) dt \right] m(P, k) \\ & < \left[ \frac{K(K-\rho)(n|a_0| - K|a_1|)}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \right. \\ & \quad \left. + \left\{ 1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \right\} \left(\frac{K+\rho}{K+r}\right)^n \right] M(P, r) \end{aligned}$$





Title Page

Contents



Page 9 of 15

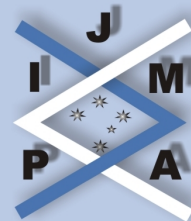
Go Back

Full Screen

Close

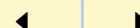
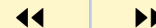
$$\begin{aligned}
 & -n \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \int_r^\rho \left( \left( \frac{t+K}{r+K} \right)^{n-1} - 1 \right) dt \right\} m(P, k) \\
 = & \left( \frac{K+\rho}{K+r} \right)^n \left[ 1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \left\{ 1 - \left( \frac{K+r}{K+\rho} \right)^n \right\} \right] M(P, r) \\
 & -n \left\{ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right\} \frac{1}{(r+K)^{n-1}} \\
 & \times \left\{ \frac{(\rho+K)^n - (r+K)^n}{n} - (\rho-r) \right\} m(P, k) \\
 = & \left( \frac{K+\rho}{K+r} \right)^n \left[ 1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \right. \\
 & \times \left. \frac{(\rho-r)}{(K+\rho) \left\{ 1 - \frac{K+r}{K+\rho} \right\}} \left\{ 1 - \left( \frac{K+r}{K+\rho} \right)^n \right\} \right] M(P, r) \\
 & - \left[ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right. \\
 & \times (r+K) \left. \left\{ \left\{ \left( \frac{\rho+K}{r+K} \right)^n - 1 \right\} - n(\rho-r) \right\} \right] m(P, k) \\
 \leq & \left( \frac{K+\rho}{K+r} \right)^n \left[ 1 - \frac{K(K-\rho)(n|a_0| - K|a_1|)n}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \left( \frac{\rho-r}{K+\rho} \right) \left( \frac{K+r}{K+\rho} \right)^{n-1} \right] M(P, r) \\
 & - \left[ \frac{(n|a_0|\rho + K^2|a_1|)(r+K)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \left\{ \left( \left( \frac{\rho+K}{r+K} \right)^n - 1 \right) - n(\rho-r) \right\} \right] m(P, K)
 \end{aligned}$$

which proves Lemma 2.3. □



Title Page

Contents



Page 10 of 15

Go Back

Full Screen

Close

### 3. Proof of the Theorems

*Proof of Theorem 1.1.* Since the polynomial  $P(z) = \sum_{j=0}^n a_j z^j$  has no zeros in  $|z| < K$ , where  $K \geq 1$ , therefore it follows that  $F(z) = P(\rho z)$  has no zero in  $|z| < \frac{K}{\rho}$ ,  $\frac{K}{\rho} \geq 1$ . Applying inequality (1.4) to the polynomial  $F(z)$ , we get

$$\text{Max}_{|z|=1} |F'(z)| \leq \frac{n}{1 + \frac{K}{\rho}} \text{Max}_{|z|=1} |F(z)|,$$

which gives

$$(3.1) \quad \text{Max}_{|z|=\rho} |P'(z)| \leq \frac{n}{\rho + K} \text{Max}_{|z|=\rho} |P(z)|.$$

Now if  $0 \leq r \leq \rho \leq K$ , then from (3.1) it follows with the help of Lemma 2.3 that

$$\begin{aligned} \text{Max}_{|z|=\rho} |P'(z)| &\leq \frac{n(K + \rho)^{n-1}}{(K + r)^n} \left[ 1 - \frac{K(K - \rho)(n|a_0| - K|a_1|)n}{(K^2 + \rho^2)n|a_0| + 2K^2\rho|a_1|} \right. \\ &\quad \times \left. \left( \frac{\rho - r}{K + \rho} \right) \left( \frac{K + r}{K + \rho} \right)^{n-1} \right] M(P, r) \\ &\quad - n \left( \frac{r + K}{\rho + K} \right) \left[ \frac{(n|a_0|\rho + K^2|a_1|)}{(\rho^2 + K^2)n|a_0| + 2K^2\rho|a_1|} \right. \\ &\quad \times \left. \left\{ \left( \left( \frac{\rho + K}{r + K} \right)^n - 1 \right) - n(\rho - r) \right\} \right] m(P, K), \end{aligned}$$

which completes the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* If  $m = \text{Min}_{|z|=K} |P(z)|$ , then  $m \leq |P(z)|$  for  $|z| = K$ , which gives  $m|\frac{z}{K}|^t \leq |P(z)|$  for  $|z| = K$ . Since all the zeros of  $P(z)$  lie in  $|z| \leq$



Title Page

Contents



Page 11 of 15

Go Back

Full Screen

Close

$K \leq 1$ , with  $t$ -fold zeros at the origin, therefore for every complex number  $\alpha$  such that  $|\alpha| < 1$ , it follows (by Rouches' Theorem for  $m > 0$ ) that the polynomial  $G(z) = P(z) + \frac{\alpha m}{K^t} z^t$  has all its zeros in  $|z| \leq K$ ,  $K \leq 1$  with  $t$ -fold zeros at the origin, so that we can write

$$(3.2) \quad G(z) = z^t H(z),$$

where  $H(z)$  is a polynomial of degree  $n - t$  having all its zeros in  $|z| \leq K$ ,  $K \leq 1$ .

From (3.2), we get

$$(3.3) \quad \frac{zG'(z)}{G(z)} = t + \frac{zH'(z)}{H(z)}.$$

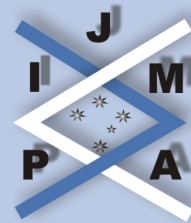
If  $z_1, z_2, \dots, z_{n-t}$  are the zeros of  $H(z)$ , then  $|z_j| \leq K \leq 1$  and from (3.3), we have

$$(3.4) \quad \begin{aligned} \operatorname{Re} \left\{ \frac{e^{i\theta} G'(e^{i\theta})}{G(e^{i\theta})} \right\} &= t + \operatorname{Re} \left\{ \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta})} \right\} \\ &= t + \operatorname{Re} \sum_{j=1}^{n-t} \frac{e^{i\theta}}{e^{i\theta} - z_j} \\ &= t + \sum_{j=1}^{n-t} \operatorname{Re} \left( \frac{1}{1 - z_j e^{-i\theta}} \right) \end{aligned}$$

for points  $e^{i\theta}$ ,  $0 \leq \theta < 2\pi$  which are not the zeros of  $H(z)$ .

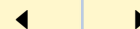
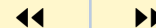
Now, if  $|w| \leq K \leq 1$ , then it can be easily verified that

$$\operatorname{Re} \left( \frac{1}{1-w} \right) \geq \frac{1}{1+K}.$$



Title Page

Contents



Page 12 of 15

Go Back

Full Screen

Close

Using this fact in (3.4), we see that

$$\begin{aligned} \left| \frac{G'(e^{i\theta})}{G(e^{i\theta})} \right| &\geq \operatorname{Re} \left( \frac{e^{i\theta} G'(e^{i\theta})}{G(e^{i\theta})} \right) \\ &= t + \sum_{j=1}^{n-t} \operatorname{Re} \left( \frac{1}{1 - z_j e^{-i\theta}} \right) \geq t + \frac{n-t}{1+K}, \end{aligned}$$

which gives,

$$(3.5) \quad |G'(e^{i\theta})| \geq \frac{n+tK}{1+K} |G(e^{i\theta})|$$

for points  $e^{i\theta}$ ,  $0 \leq \theta < 2\pi$  which are not the zeros of  $G(z)$ . Since inequality (3.5) is trivially true for points  $e^{i\theta}$ ,  $0 \leq \theta < 2\pi$  which are the zeros of  $P(z)$ , it follows that

$$(3.6) \quad |G'(z)| \geq \frac{n+tK}{1+K} |G(z)| \quad \text{for } |z| = 1.$$

Replacing  $G(z)$  by  $P(z) + \frac{\alpha m}{K^t} z^t$  in (3.6), then we get

$$(3.7) \quad \left| P'(z) + \frac{\alpha t m}{K^t} z^{t-1} \right| \geq \frac{n+tK}{1+K} \left| P(z) + \frac{\alpha m}{K^t} z^t \right| \quad \text{for } |z| = 1$$

and for every  $\alpha$  with  $|\alpha| < 1$ . Choosing the argument of  $\alpha$  such that

$$\left| P(z) + \frac{\alpha m}{K^t} z^t \right| = |P(z)| + |\alpha| \frac{m}{K^t} \quad \text{for } |z| = 1,$$

it follows from (3.7) that

$$|P'(z)| + \frac{t|\alpha|m}{K^t} \geq \frac{n+tK}{1+K} \left[ |P(z)| + |\alpha| \frac{m}{K^t} \right] \quad \text{for } |z| = 1.$$

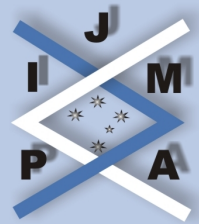
Letting  $|\alpha| \rightarrow 1$ , we obtain

$$\begin{aligned} |P'(z)| &\geq \frac{n+tK}{1+K} |P(z)| + \left[ \frac{n+tK}{1+K} - t \right] \frac{m}{K^t} \\ &= \frac{n+tK}{1+K} |P(z)| + \frac{n-t}{1+K} \frac{m}{K^t} \quad \text{for } |z| = 1. \end{aligned}$$

This implies

$$\text{Max}_{|z|=1} |P'(z)| \geq \frac{n+tK}{1+K} \text{Max}_{|z|=1} |P(z)| + \frac{n-t}{(1+K)K^t} \text{Min}_{|z|=K} |P(z)|$$

which is the desired result. □



Title Page

Contents



Page 13 of 15

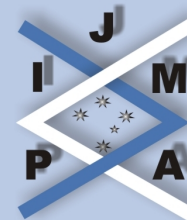
Go Back

Full Screen

Close

## References

- [1] A. AZIZ AND Q.M. DAWOOD, Inequalities for a polynomial and its derivative, *J. Approx. Theory*, **54** (1988), 306–313.
- [2] A. AZIZ AND W.M. SHAH, Inequalities for a polynomial and its derivative, *Math. Inequal. and Applics.*, **7**(3) (2004), 379–391.
- [3] A. AZIZ AND B.A. ZARGAR, Inequalities for a polynomial and its derivative, *Math. Inequal. and Applics.*, **1**(4) (1998), 543–550.
- [4] S. BERNSTEIN, Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, *Mémoires de l'Académie Royale de Belgique*, **4** (1912), 1–103.
- [5] S. BERNSTEIN, Leçons sur les propriétés extrémales et la meilleure d'une fonctions réelle, Paris 1926.
- [6] K.K. DEWAN AND A. MIR, On the maximum modulus of a polynomial and its derivatives, *International Journal of Mathematics and Mathematical Sciences*, **16** (2005), 2641–2645.
- [7] N.K. GOVIL, Q.I. RAHMAN AND G. SCHMEISSER, On the derivative of a polynomial, *Illinois J. Math.*, **23** (1979), 319–329.
- [8] P.D. LAX, Proof of a conjecture of P. Erdős on the derivative of a polynomial, *Bull. Amer. Math. Soc. (N.S)*, **50** (1944), 509–513.
- [9] M.A. MALIK, On the derivative of a polynomial, *J. London Math. Soc.*, **1** (1969), 57–60.
- [10] A.C. SCHAFFER, Inequalities of A. Markoff and S. Bernstein for polynomials and related functions, *Bull. Amer. Math. Soc.*, **47** (1941), 565–579.



Inequalities for the  
Maximum Modulus

A. Aziz and B.A. Zargar

vol. 8, iss. 2, art. 37, 2007

Title Page

Contents



Page 14 of 15

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

- [11] G.V. MILOVANOVIĆ, D.S. MITRINOVIĆ AND Th.M. RASSIAS, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Scientific, Singapore, 1994.
- [12] P. TURÁN, Über die Ableitung von Polynomen, *Compositio Math.*, **7** (1939-40), 89–95.



---

**Inequalities for the  
Maximum Modulus**

A. Aziz and B.A. Zargar

vol. 8, iss. 2, art. 37, 2007

---

Title Page

Contents



Page 15 of 15

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756