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A NOTE ON ESTIMATES OF DIAGONAL ELEMENTS OF THE INVERSE OF DIAGONALLY DOMINANT TRIDIAGONAL MATRICES

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ABSTRACT. In this note we show how to improve some recent upper and lower bounds for the elements of the inverse of diagonally dominant tridiagonal matrices. In particular, a technique described by [R. Peluso, and T. Politi, Some improvements on two-sided bounds on the inverse of diagonally dominant tridiagonal matrices, Lin. Alg. Appl. Vol. 330 (2001) 1-14], is used to obtain better bounds for the diagonal elements.

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1. INTRODUCTION

It is common in numerical analysis to find problems which are modeled by means of a tridiagonal matrix: this is, for example, the case of some kinds of discretization of partial differential equations and finite difference methods for boundary value problems. In some of these situations, the inverse of these tridiagonal matrices plays an important role, as, for example, when preconditioners for iterative solvers are needed. The available literature for this subject is very rich and we cite the survey by Meurant [5] for a list of references.

The aim of this paper is to provide good estimates for the elements of the inverse of a diagonally dominant tridiagonal matrix A. In this context, Nabben [6] presented lower and upper bounds for the diagonal entries which depend on the entries of A and bounds for the off-diagonal elements depending on the diagonal ones. A few years later, Peluso and Politi [7] improved the bounds for the diagonal elements by exploiting the sign of the quantities involved. Recently Liu et al. [4] found lower and upper bounds for the off-diagonal elements which are not always sharper than those previously found; they also presented an estimate for the diagonal elements.

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In this paper we propose improved bounds for the diagonal elements of the inverse; several numerical examples are also presented to confirm the good performance of the estimates. An application to preconditioning is also shown.

Let us introduce some notations; consider the following real tridiagonal matrix of order n, with $n \geq 3$,

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}$$

with $a_i \neq 0$, for $i = 1, \dots, n$; assume that A is *row diagonally dominant*, i.e.

$$|a_i| \geq |b_i| + |c_{i-1}|, \qquad i = 1, \dots, n, \qquad c_0 = b_n = 0.$$

It is natural to suppose that $c_i, b_i \neq 0$, for $i = 1, \dots, n-1$, as if one of these elements is 0 the problem can be reduced to smaller subproblems, see e.g. [1].

Let $C = A^{-1} = \{c_{i,j}\}$ be the inverse of A.

In the following we will use some relations derived from the identity $AC = I$; indeed, for the diagonal elements of C it results that

$$c_{j-1}c_{j-1,j} + a_j c_{j,j} + b_j c_{j+1,j} = 1, \qquad j = 2, \dots, n-1 \tag{1.1}$$

while for $j \geq 3$ and $i = 1, \dots, j$ the j-th equation of the system $AC = I$ reads

$$c_{i,j} = -\frac{a_{i-1}c_{i-1,j} + c_{i-2}c_{i-2,j}}{b_{i-1}}. \tag{1.2}$$

If $b_i < 0$, $c_i < 0$ and $a_i > 0$, and $a_1 > -b_1$ or $a_n > -c_{n-1}$, then A is an M-matrix. In this case $c_{i,j} > 0$, $\forall i, j$.

The paper is organized as follows: in Section 2 we recall a collection of known results, while in Section 3 we exploit some results described in Section 2 to improve the bounds for diagonal entries, and in Section 4 some numerical examples are shown.

2. SOME KNOWN RESULTS

The technique common to [6], [7] and [4] is bounding $|c_{i,j}|$, $i \neq j$, as a function of diagonal elements $|c_{j,j}|$ and providing suitable bounds for these.

For a review of the most recent resulting bounds presented in [4], we define the quantities

$$\xi_i = \frac{|b_i|}{|a_i| - |\alpha_{i-1}||c_{i-1}|}; \qquad m_i = \frac{|b_i|}{|a_i| + |\alpha_{i-1}||c_{i-1}|};$$

$$\lambda_i = \frac{|c_{i-1}|}{|a_i| - |\beta_{i+1}||b_i|}; \qquad n_i = \frac{|c_{i-1}|}{|a_i| + |\beta_{i-1}||b_i|},$$

where

$$\alpha_i = \frac{b_i}{p_i}, \qquad i = 1, \dots, n-1$$

$$p_i = a_i - \alpha_{i-1}c_{i-1}, \quad q_i = \beta_{i+1}b_i, \quad i = 1, \dots, n$$

$$\beta_i = \frac{c_{i-1}}{a_i - q_i}, \qquad i = n, \dots, 2,$$

provided that $\alpha_0 = \beta_{n+1} = 0$.

Using these quantities the following bounds were derived in [4].

Theorem 2.1 ([4]). *If* $|a_i| - |\beta_{i+1}||b_i| > 0$ *for* $i = 2, \dots, n$, *and* $|a_i| - |\alpha_{i-1}||c_{i-1}| > 0$ *for* $i = 1, \dots, n-1$, *then the following bounds hold for the elements of* C*:*

$$n_i|c_{i-1,j}| \le |c_{i,j}| \le \lambda_i|c_{i-1,j}|, \qquad i = j+1, \dots, n, \ j = 1, \dots, n-1, \tag{2.1}$$

$$m_i|c_{i+1,j}| \le |c_{i,j}| \le \xi_i|c_{i+1,j}|, \qquad i = 1, \dots, j-1, \ j = 2, \dots, n, \tag{2.2}$$

$$|c_{j,j}| \prod_{k=j+1}^{i} n_k \le |c_{i,j}| \le |c_{j,j}| \prod_{k=j+1}^{i} \lambda_k, \qquad \text{for } i > j, \tag{2.3}$$

$$|c_{j,j}| \prod_{k=i}^{j-1} m_k \le |c_{i,j}| \le |c_{j,j}| \prod_{k=i}^{j-1} \xi_k, \qquad \text{for } i < j, \tag{2.4}$$

while for the diagonal entries

$$\frac{1}{|a_j| + \xi_{j-1}|c_{j-1}| + \lambda_{j+1}|b_j|} \le |c_{j,j}| \le \frac{1}{|a_j| - \xi_{j-1}|c_{j-1}| - \lambda_{j+1}|b_j|}. \tag{2.5}$$

Remark 1. The hypotheses of Theorem 2.1 hold when A is a strictly diagonally dominant or an irreducibly diagonally dominant matrix [4].

We recall the results presented in [7], for which we need the following definitions:

$$\tau_i = \frac{|b_i|}{|a_i| - |c_{i-1}|} \qquad \delta_i = \frac{|b_i|}{|a_i| + |c_{i-1}|} \qquad i = 1, \dots, n-1,$$

$$\omega_i = \frac{|c_{i-1}|}{|a_i| - |b_i|} \qquad \gamma_i = \frac{|c_{i-1}|}{|a_i| + |b_i|} \qquad i = 2, \dots, n,$$

$$s_j = \begin{cases} \tau_{j-1} & \text{if } a_{j-1}a_jb_{j-1}c_{j-1} < 0; \\ -\delta_{j-1} & \text{if } a_{j-1}a_jb_{j-1}c_{j-1} > 0, \end{cases}$$

$$t_j = \begin{cases} \omega_{j+1} & \text{if } a_{j+1}a_jb_jc_j < 0; \\ -\gamma_{j+1} & \text{if } a_{j+1}a_jb_jc_j > 0, \end{cases}$$

$$f_j = \begin{cases} -\tau_{j-1} & \text{if } a_{j-1}a_jb_{j-1}c_{j-1} > 0; \\ \delta_{j-1} & \text{if } a_{j-1}a_jb_{j-1}c_{j-1} < 0, \end{cases}$$

$$g_j = \begin{cases} -\omega_{j+1} & \text{if } a_{j+1}a_jb_jc_j > 0; \\ \gamma_{j+1} & \text{if } a_{j+1}a_jb_jc_j < 0. \end{cases}$$

Theorem 2.2 ([7]). *Let* A *be a nonsingular tridiagonal matrix. If* A *is row diagonally dominant, then the following bounds hold for the elements of* C*:*

$$|c_{j,j}| \prod_{k=j+1}^{i} \delta_k \le |c_{i,j}| \le |c_{j,j}| \prod_{k=j+1}^{i} \tau_k, \qquad \text{for } i > j; \tag{2.6}$$

$$|c_{j,j}| \prod_{k=i}^{j-1} \gamma_k \le |c_{i,j}| \le |c_{j,j}| \prod_{k=i}^{j-1} \omega_k, \qquad \text{for } i < j, \tag{2.7}$$

while for the diagonal entries

$$\frac{1}{|a_j| + s_j|c_{j-1}| + t_j|b_j|} \le |c_{j,j}| \le \frac{1}{|a_j| + f_j|c_{j-1}| + g_j|b_j|}. \tag{2.8}$$

Remark 2. There are few remarks for both the theorems recalled above, which have not been stressed so far.

From both theorems it is clear that for the off-diagonal elements we have $|c_{i,j}| \le |c_{i-1,j}|$, since all the coefficients involved are less than 1; this leads to the well known result about the decreasing pattern of the elements of the inverse of a banded matrix, see e.g. [3].

From the lower bounds for the diagonal entries it is clear that $c_{i,i} \ge \frac{1}{a_{i,i}}$, which was a lower bound presented by Meurant [5] in the special case of diagonally dominant M-matrices, and so the estimates in (2.8) are more accurate than those in [5], although they are more costly to evaluate.

Remark 3. For $i < j$, the following inequalities hold (see [4]):

$$\begin{array}{ll} m_k \ge \delta_k, & k = i, \dots, j-1; \\ \xi_k \le \tau_k, & k = i, \dots, j-1, \end{array} \tag{2.9}$$

and, for $i > j$:

$$\begin{array}{ll} n_k \ge \gamma_k, & k = j+1, \dots, i; \\ \lambda_k \le \omega_k, & k = j+1, \dots, i, \end{array} \tag{2.10}$$

hence the bounds for the extradiagonal elements reported in Theorem 2.1 improve those given in 2.2.

2.1. **Special cases.** In the extensive survey about the inverse of tridiagonal matrices [5], Meurant considered also the special case in which one is interested in the inverse C of a *Toeplitz* matrix T_a, having a on the main diagonal and -1 on the other two diagonals. The provided upper bounds share the common idea of [4] and [7], that is $|c_{i,j}|$, $i \ne j$, is bounded as a function of $|c_{j,j}|$ and suitable bounds are presented for the diagonal entries.

We now recall that result:

Theorem 2.3. [5] *If* $a > 2$ *then*

$$(T_a^{-1})_{i,j} < r_-^{j-i}(T_a^{-1})_{i,i}, \quad \forall i, \forall j \ge i \tag{2.11}$$

$$(T_a^{-1})_{i,j} < \frac{r_-^{j-i+1}}{1-r}, \quad \forall i, \forall j \ge i+1 \tag{2.12}$$

for $r_\pm = \frac{a \pm \sqrt{(a^2-4))}}{2}$; $r = \frac{r_-}{r_+}$.

3. Lower and Upper Bounds for the Diagonal Elements

The following example motivated our investigation. We consider the tridiagonal matrix

$$A = \begin{bmatrix} 4 & 2 & & & \\ -2 & 4 & 2 & & \\ & 2 & 4 & 2 & \\ & & -2 & -4 & -2 \\ & & & -2 & 4 \end{bmatrix}$$

for which the hypotheses of Theorem 2.1 are true, and compare the bounds given by [7] and by [4]: to consider the bounds of [4], we construct the matrices U and L whose (i,j) entry represents the upper, respectively the lower, bound in (2.3) and in (2.4), whereas the (j, j) entry is the upper, respectively the lower, bound in (2.5). We use the same notation for representing the bounds given in [7].

In Table 3.1 the maximum errors on the upper and lower bounds are represented. From Table

	$\max_{i,j}\{U_{i,j} - \lvert c_{i,j}\rvert\}$	$\max_{i,j}\{\lvert c_{i,j}\rvert - L_{i,j}\}$
(2.6),(2.7),(2.8)	0.4706	0.0846
(2.3),(2.4),(2.5)	0.6366	0.1213
	$\max_{j}\{U_{j,j} - \lvert c_{j,j}\rvert\}$	$\max_{j}\{\lvert c_{j,j}\rvert - L_{j,j}\}$
(2.8)	0.1912	0.0353
(2.5)	0.6366	0.1213

Table 3.1:

3.1 we observe that the bounds by Liu et al. have a maximum error greater than that by Peluso et al.; the reason is that the bounds derived in [4] do not improve those on the diagonal elements hence the global bounds are not better for all the elements of the matrix. Looking inside the bound matrices we observe that the bounds improve for 40% of the entries and for 20% of diagonal elements when using (2.3), (2.4) and (2.5) in place of (2.3), (2.4), (3.1).

Our aim is to obtain sharper two-sided bounds for the diagonal elements of C, exploiting the signs of its entries, using exactly the same technique as in [7]. We now give the main result of the present paper.

Theorem 3.1. *Let A be a nonsingular tridiagonal matrix and $C = A^{-1}$. If A is row diagonally dominant then*

$$\frac{1}{|a_j| + s_j|c_{j-1}| + t_j|b_j|} \leq |c_{j,j}| \leq \frac{1}{|a_j| + f_j|c_{j-1}| + g_j|b_j|}, \quad j = 1, \ldots, n, \tag{3.1}$$

where

$$s_j = \begin{cases} \xi_{j-1} & \text{if } a_{j-1}a_j b_{j-1}c_{j-1} < 0; \\ -m_{j-1} & \text{if } a_{j-1}a_j b_{j-1}c_{j-1} > 0, \end{cases}$$

$$t_j = \begin{cases} \lambda_{j+1} & \text{if } a_{j+1}a_j b_j c_j < 0; \\ -n_{j+1} & \text{if } a_{j+1}a_j b_j c_j > 0, \end{cases}$$

$$f_j = \begin{cases} -\xi_{j-1} & \text{if } a_{j-1}a_j b_{j-1}c_{j-1} > 0; \\ m_{j-1} & \text{if } a_{j-1}a_j b_{j-1}c_{j-1} < 0, \end{cases}$$

$$g_j = \begin{cases} -\lambda_{j+1} & \text{if } a_{j+1}a_j b_j c_j > 0; \\ n_{j+1} & \text{if } a_{j+1}a_j b_j c_j < 0, \end{cases}$$

Proof. We fix j and define $\mu_j = c_{j-1,j}/c_{j,j}$ and $\rho_j = c_{j+1,j}/c_{j,j}$; if we use these definitions in (1.1) we get

$$c_{j,j} = \frac{1}{a_j + \mu_j c_{j-1} + \rho_j b_j} = \frac{1}{a_j \left(1 + \mu_j \frac{c_{j-1}}{a_j} + \rho_j \frac{b_j}{a_j}\right)}$$

and

$$|c_{j,j}| = \frac{1}{|a_j|\left(1+\varepsilon_1|\mu_j|\frac{|c_{j-1}|}{|a_j|}+\varepsilon_2|\rho_j|\frac{|b_j|}{|a_j|}\right)} \tag{3.2}$$

with $\varepsilon_1 = \operatorname{sign}(\mu_j c_{j-1} a_j)$ and $\varepsilon_2 = \operatorname{sign}(\rho_j b_j a_j)$.

For $i = j-1$, (2.2) gives

$$m_{j-1}|c_{j,j}| \le |c_{j-1,j}| \le \xi_{j-1}|c_{j,j}|,$$

that is

$$m_{j-1} \le |\mu_j| \le \xi_{j-1}. \tag{3.3}$$

If $\varepsilon_1 > 0$, then $m_{j-1}|c_{j-1}| \le \varepsilon_1|\mu_j||c_{j-1}| \le \xi_{j-1}|c_{j-1}|$, while if $\varepsilon_1 < 0$, then $-\xi_{j-1}|c_{j-1}| \le \varepsilon_1|\mu_j||c_{j-1}| \le -m_{j-1}|c_{j-1}|$ and from these relations we get the expression for s_j.

It remains to show that $\varepsilon_1 = \operatorname{sign}(a_j a_{j-1} b_{j-1} c_{j-1})$.

The equation (1.2) for $i = j$ gives

$$\frac{c_{j,j}}{c_{j-1,j}} = -\frac{a_{j-1}}{b_{j-1}} - \frac{c_{j-2}}{b_{j-1}}\frac{c_{j-2,j}}{c_{j-1,j}}.$$

Since $|c_{j-2,j}| \le |c_{j-1,j}|$ and $|a_{j-1}| > |c_{j-2}|$, then

$$-\operatorname{sign}(a_{j-1}b_{j-1}) = \operatorname{sign}(c_{j,j}c_{j-1,j}) = \operatorname{sign}(\mu_j)$$

from which we have the expression for ε_1.

Using exactly the same technique, we may work on the upper bounds and discuss the influence of ε_2, as in [7]. □

Again, as in [7], we can write explicit bounds when A is an M-matrix.

Remark 4. If A is an M-matrix, then the bounds proved in Theorem 3.1 become

$$\frac{1}{a_j - m_{j-1}|c_{j-1}| - n_{j+1}|b_j|} \le c_{j,j} \le \frac{1}{a_j - \xi_{j-1}|c_{j-1}| - \lambda_{j+1}|b_j|}, \quad j = 1, \dots, n.$$

We note that the upper bound is the same as that in [4] but the lower bound is sharper.

Remark 5. As a consequence of (2.9) and (2.10), the bounds proved in Theorem 3.1 are sharper (or not worse) than those shown in [4].

4. NUMERICAL EXAMPLES

As one may expect, all the lower and upper bounds presented above do not depend on the dimension of the matrices involved. All the numerical experiments we did showed in fact that varying the dimension of the matrix does not influence the accuracy of the estimates. For this reason, in this section we consider problems of small size, but we stress that our results work well also for much larger problems.

Example 4.1. Let

$$A = \begin{bmatrix} -34 & -13.4 & & & & \\ -2.2 & 3 & 0.5 & & & \\ & 3.3 & 45 & -2.3 & & \\ & & 2.1 & -22 & 0.6 & \\ & & & 3 & 15 & 0.22 \\ & & & & 1.3 & 42 \end{bmatrix}$$

and let B be the inverse of A, computed by using the Matlab function `inv`. For considering the bounds (2.3), (2.4), (2.5) of Liu et al. [4], we construct the matrices U and L whose (i,j) entry represents the upper, respectively the lower, bound in (2.3) and in (2.4), whereas the (j, j) entry is the upper, respectively the lower, bound in (2.5). We use the same notation for representing the bounds obtained by using (2.3), (2.4) with the diagonal bounds (3.1).

	$\max_{i,j}\{U_{i,j} - \|c_{i,j}\|\}$	$\max_{i,j}\{\|c_{i,j}\| - L_{i,j}\}$
(2.3), (2.4), (3.1)	$2.4119e - 003$	$4.0993e - 004$
(2.3), (2.4), (2.5)	$2.1600e - 001$	$1.0290e - 001$
	$\max_{j}\{U_{j,j} - \|c_{j,j}\|\}$	$\max_{j}\{\|c_{j,j}\| - L_{j,j}\}$
(2.3), (2.4), (3.1)	$1.7895e - 004$	$2.4253e - 005$
(2.3), (2.4), (2.5)	$2.1600e - 001$	$4.9282e - 003$

Table 4.1: Comparison with the Liu et al.'s upper bounds

The entries of Table 4.1 show an improvement in the accuracy of the estimates of at least two orders of magnitude.

Example 4.2. An important example of strictly diagonally dominant tridiagonal matrices is

$$T_4 = \begin{bmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & \ddots & \ddots & \\ & & & -1 & 4 \\ & & & & \end{bmatrix};$$

it arises when discretizing the Laplace operator on a rectangular domain with Dirichlet boundary conditions: it actually represents the structure of the diagonal blocks of the discretization of the Laplacian obtained by applying a five-point finite difference.

In [2], Benzi and Golub used this matrix to test an approximate inverse; they stress that an approximate inverse of T_4 is needed in the initial step of an incomplete Cholesky factorization for the two-dimensional model problem. By applying the equations (2.3), (2.4), (3.1) we are able to obtain good bounds for the generic entry of T_4^{-1}. We define the matrices U and L as in the previous example and, with a very naïve idea, we consider the matrix $M := \frac{1}{2}(U + L)$ as an approximant to T_4^{-1}; we use a matrix T_4 of dimension 100×100.

As in [2], we use the condition number cond of the matrix MT_4 to test the quality of the approximation. We notice that $\operatorname{cond}(MT_4) = 1.0736$, from which we may state that M would be an effective preconditioner.

As mentioned in [2], it is well known that, due to the decay of the elements of the inverse, considering only the tridiagonal approximations provides good results. We considered only the tridiagonal part of M, say M_1, to obtain $\operatorname{cond}(M_1T_4) = 1.2185$.

With the same matrix T_4, the upper bounds in (2.3) and (2.4) are sharp when we use (3.1) for the diagonal entries. We in fact observed that

$$\max_{i,j}\{U_{i,j} - |c_{i,j}|\} < 10^{-16},$$

where U_{ij} and c_{ij} represent, respectively, the (i, j) entry of matrices U, of the upper bounds (2.3), (2.4) and (3.1), and T_4^{-1}.

Example 4.3. We consider the Toeplitz matrix T_4 of dimension 8×8; we compare the upper bounds presented in this paper with those proposed by Meurant [5].

	$\max_{i;j\geq i}\{U_{i,j} - \vert c_{i,j}\vert\}$
(2.3), (2.4), (3.1)	$8.6736e-019$
(2.11)	$1.2780e-007$
	$\max_{i;j\geq i+1}\{U_{i,j} - \vert c_{i,j}\vert\}$
(2.3), (2.4), (3.1)	$2.8709e-017$
(2.12)	$5.7654e-003$

Table 4.2: Comparison with the Meurant's upper bounds

The results in Table 4.2 show an impressive improvement when using the bounds (2.3), (2.4), (3.1).

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