

ON OPEN PROBLEMS OF F. QI

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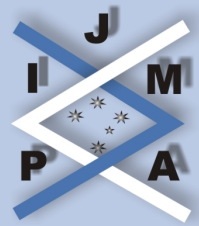
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Abstract: In this paper, we give a complete answer to Problem 1 and a partial answer to Problem 2 posed by F. Qi in [2] and we propose an open problem.

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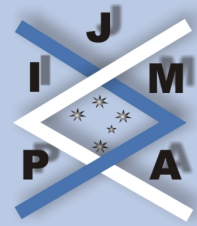
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1. Introduction

Before, we state our results, for our own convenience, we introduce the following notations:

$$(1.1) \quad [0, \infty)^n \triangleq \underbrace{[0, \infty) \times [0, \infty) \times \dots \times [0, \infty)}_{n \text{ times}}$$

and

$$(1.2) \quad (0, \infty)^n \triangleq \underbrace{(0, \infty) \times (0, \infty) \times \dots \times (0, \infty)}_{n \text{ times}}$$

for $n \in \mathbb{N}$, where \mathbb{N} denotes the set of all positive integers.

In [2], F. Qi proved the following:

Theorem A. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, inequality

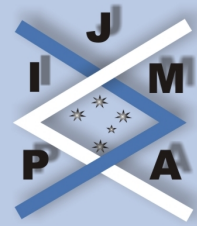
$$(1.3) \quad \frac{e^2}{4} \sum_{i=1}^n x_i^2 \leq \exp \left(\sum_{i=1}^n x_i \right)$$

is valid. Equality in (1.3) holds if $x_i = 2$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1.3) is the best possible.

Theorem B. Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. Then

$$(1.4) \quad \frac{e^2}{4} \sum_{i=1}^{\infty} x_i^2 \leq \exp \left(\sum_{i=1}^{\infty} x_i \right).$$

Equality in (1.4) holds if $x_i = 2$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^2}{4}$ in (1.4) is the best possible.



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In the same paper, F. Qi posed the following two open problems:

Problem 1.1. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, determine the best possible constants $\alpha_n, \lambda_n \in \mathbb{R}$ and $\beta_n > 0, \mu_n < \infty$ such that

$$(1.5) \quad \beta_n \sum_{i=1}^n x_i^{\alpha_n} \leq \exp \left(\sum_{i=1}^n x_i \right) \leq \mu_n \sum_{i=1}^n x_i^{\lambda_n}.$$

Problem 1.2. What is the integral analogue of the double inequality (1.5)?

Recently, Huan-Nan Shi gave a partial answer in [3] to Problem 1.1. The main purpose of this paper is to give a complete answer to this problem. Also, we give a partial answer to Problem 1.2. The method used in this paper will be quite different from that in the proofs of Theorem 1.1 of [2] and Theorem 1 of [3]. For some related results, we refer the reader to [1]. We will prove the following results.

Theorem 1.1. Let $p \geq 1$ be a real number. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, the inequality

$$(1.6) \quad \frac{e^p}{p^p} \sum_{i=1}^n x_i^p \leq \exp \left(\sum_{i=1}^n x_i \right)$$

is valid. Equality in (1.6) holds if $x_i = p$ for some given $1 \leq i \leq n$ and $x_j = 0$ for all $1 \leq j \leq n$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (1.6) is the best possible.

Theorem 1.2. Let $0 < p \leq 1$ be a real number. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, the inequality

$$(1.7) \quad n^{p-1} \frac{e^p}{p^p} \sum_{i=1}^n x_i^p \leq \exp \left(\sum_{i=1}^n x_i \right)$$



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is valid. Equality in (1.7) holds if $x_i = \frac{p}{n}$ for all $1 \leq i \leq n$. Thus, the constant $n^{p-1} \frac{e^p}{p^p}$ in (1.7) is the best possible.

Theorem 1.3. Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$ and $p \geq 1$ be a real number. Then

$$(1.8) \quad \frac{e^p}{p^p} \sum_{i=1}^{\infty} x_i^p \leq \exp \left(\sum_{i=1}^{\infty} x_i \right).$$

Equality in (1.8) holds if $x_i = p$ for some given $i \in \mathbb{N}$ and $x_j = 0$ for all $j \in \mathbb{N}$ with $j \neq i$. Thus, the constant $\frac{e^p}{p^p}$ in (1.8) is the best possible.

Remark 1. In general, we cannot find $0 < \mu_n < \infty$ and $\lambda_n \in \mathbb{R}$ such that

$$\exp \left(\sum_{i=1}^n x_i \right) \leq \mu_n \sum_{i=1}^n x_i^{\lambda_n}.$$

Proof. We suppose that there exists $0 < \mu_n < \infty$ and $\lambda_n \in \mathbb{R}$ such that

$$\exp \left(\sum_{i=1}^n x_i \right) \leq \mu_n \sum_{i=1}^n x_i^{\lambda_n}.$$

Then for $(x_1, 1, \dots, 1)$, we obtain as $x_1 \rightarrow +\infty$,

$$1 \leq e^{1-n} \mu_n (n-1 + x_1^{\lambda_n}) e^{-x_1} \rightarrow 0.$$

This is a contradiction. □



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Theorem 1.4. Let $p > 0$ be a real number, $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$ such that $0 < x_i \leq p$ for all $1 \leq i \leq n$. Then the inequality

$$(1.9) \quad \exp\left(\sum_{i=1}^n x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^n x_i^{-p}$$

is valid. Equality in (1.9) holds if $x_i = p$ for all $1 \leq i \leq n$. Thus, the constant $\frac{p^p}{n} e^{np}$ is the best possible.

Remark 2. Let $p > 0$ be a real number, $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$ such that $0 < x_i \leq p$ for all $1 \leq i \leq n$. Then

(i) if $0 < p \leq 1$, we have

$$(1.10) \quad n^{p-1} \frac{e^p}{p^p} \sum_{i=1}^n x_i^p \leq \exp\left(\sum_{i=1}^n x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^n x_i^{-p};$$

(ii) if $p \geq 1$, we have

$$(1.11) \quad \frac{e^p}{p^p} \sum_{i=1}^n x_i^p \leq \exp\left(\sum_{i=1}^n x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^n x_i^{-p}.$$

Remark 3. Taking $p = 2$ in Theorems 1.1 and 1.3 easily leads to Theorems A and B respectively.

Remark 4. Inequality (1.6) can be rewritten as either

$$(1.12) \quad \frac{e^p}{p^p} \sum_{i=1}^n x_i^p \leq \prod_{i=1}^n e^{x_i}$$



or

$$(1.13) \quad \frac{e^p}{p^p} \|x\|_p^p \leq \exp \|x\|_1,$$

where $x = (x_1, x_2, \dots, x_n)$ and $\|\cdot\|_p$ denotes the p -norm.

Remark 5. Inequality (1.8) can be rewritten as

$$(1.14) \quad \frac{e^p}{p^p} \sum_{i=1}^{\infty} x_i^p \leq \prod_{i=1}^{\infty} e^{x_i}$$

which is equivalent to inequality (1.12) for $x = (x_1, x_2, \dots) \in [0, \infty)^\infty$.

Remark 6. Taking $x_i = \frac{1}{i}$ for $i \in \mathbb{N}$ in (1.6) and rearranging gives

$$(1.15) \quad p - p \ln p + \ln \left(\sum_{i=1}^n \frac{1}{i^p} \right) \leq \sum_{i=1}^n \frac{1}{i}.$$

Taking $x_i = \frac{1}{i^s}$ for $i \in \mathbb{N}$ and $s > 1$ in (1.8) and rearranging gives

$$(1.16) \quad p - p \ln p + \ln \left(\sum_{i=1}^{\infty} \frac{1}{i^{ps}} \right) = p - p \ln p + \ln \zeta(ps) \leq \sum_{i=1}^{\infty} \frac{1}{i^s} = \zeta(s),$$

where ζ denotes the well-known Riemann Zêta function.

In the following, we give a partial answer to Problem 1.2.

Theorem 1.5. Let $0 < p \leq 1$ be a real number, and let f be a continuous function on $[a, b]$. Then the inequality

$$(1.17) \quad \frac{e^p}{p^p} (b-a)^{p-1} \int_a^b |f(x)|^p dx \leq \exp \left(\int_a^b |f(x)| dx \right)$$

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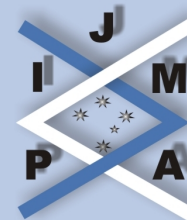
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is valid. Equality in (1.17) holds if $f(x) = \frac{p}{b-a}$. Thus, the constant $\frac{e^p}{p^p} (b-a)^{p-1}$ in (1.17) is the best possible.

Theorem 1.6. Let $x > 0$. Then

$$(1.18) \quad \Gamma(x) \leq \frac{2^{x+1}x^{x-1}}{e^x}$$

is valid, where Γ denotes the well-known Gamma function.

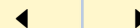
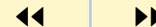


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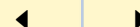
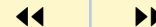
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2. Lemmas

Lemma 2.1. For $x \in [0, \infty)$ and $p > 0$, the inequality

$$(2.1) \quad \frac{e^p}{p^p} x^p \leq e^x$$

is valid. Equality in (2.1) holds if $x = p$. Thus, the constant $\frac{e^p}{p^p}$ in (2.1) is the best possible.

Proof. Letting $f(x) = p \ln x - x$ on the set $(0, \infty)$, it is easy to obtain that the function f has a maximal point at $x = p$ and the maximal value equals $f(p) = p \ln p - p$. Then, we obtain (2.1). It is clear that the inequality (2.1) also holds at $x = 0$. \square

Lemma 2.2. Let $p > 0$ be a real number. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $n \geq 2$, we have:

(i) If $p \geq 1$, then the inequality

$$(2.2) \quad \sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p$$

is valid.

(ii) If $0 < p \leq 1$, then inequality

$$(2.3) \quad n^{p-1} \sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p$$

is valid.



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Proof. (i) For the proof, we use mathematical induction. First, we prove (2.2) for $n = 2$. We have for any $(x_1, x_2) \neq (0, 0)$

$$(2.4) \quad \frac{x_1}{x_1 + x_2} \leq 1 \quad \text{and} \quad \frac{x_2}{x_1 + x_2} \leq 1.$$

Then, by $p \geq 1$ we get

$$(2.5) \quad \left(\frac{x_1}{x_1 + x_2} \right)^p \leq \frac{x_1}{x_1 + x_2} \quad \text{and} \quad \left(\frac{x_2}{x_1 + x_2} \right)^p \leq \frac{x_2}{x_1 + x_2}.$$

By addition from (2.5), we obtain

$$\left(\frac{x_1}{x_1 + x_2} \right)^p + \left(\frac{x_2}{x_1 + x_2} \right)^p \leq \frac{x_1}{x_1 + x_2} + \frac{x_2}{x_1 + x_2}.$$

So,

$$(2.6) \quad x_1^p + x_2^p \leq (x_1 + x_2)^p.$$

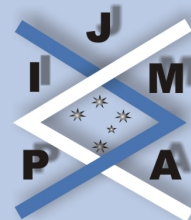
It is clear that inequality (2.6) holds also at the point $(0, 0)$.

Now we suppose that

$$(2.7) \quad \sum_{i=1}^n x_i^p \leq \left(\sum_{i=1}^n x_i \right)^p$$

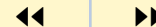
and we prove that

$$(2.8) \quad \sum_{i=1}^{n+1} x_i^p \leq \left(\sum_{i=1}^{n+1} x_i \right)^p.$$



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We have by (2.6)

$$(2.9) \quad \left(\sum_{i=1}^{n+1} x_i \right)^p = \left(\sum_{i=1}^n x_i + x_{n+1} \right)^p \geq \left(\sum_{i=1}^n x_i \right)^p + x_{n+1}^p$$

and by (2.7) and (2.9), we obtain

$$(2.10) \quad \sum_{i=1}^{n+1} x_i^p = \sum_{i=1}^n x_i^p + x_{n+1}^p \leq \left(\sum_{i=1}^n x_i \right)^p + x_{n+1}^p \leq \left(\sum_{i=1}^{n+1} x_i \right)^p.$$

Then for all $n \geq 2$, (2.2) holds.

(ii) For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$, $0 < p \leq 1$ and $n \geq 2$, we have

$$(2.11) \quad \left(\sum_{i=1}^n x_i \right)^p = \left(\sum_{i=1}^n n \frac{x_i}{n} \right)^p.$$

By using the concavity of the function $x \mapsto x^p$ ($x \geq 0$, $0 < p \leq 1$), we obtain from (2.11)

$$(2.12) \quad \left(\sum_{i=1}^n x_i \right)^p = \left(\sum_{i=1}^n n \frac{x_i}{n} \right)^p \geq \sum_{i=1}^n \frac{n^p x_i^p}{n} = n^{p-1} \sum_{i=1}^n x_i^p.$$

Hence (2.3) holds. □



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3. Proofs of the Theorems

We are now in a position to prove our theorems.

Proof of Theorem 1.1. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $p \geq 1$, we put $x = \sum_{i=1}^n x_i$. Then by (2.1), we have

$$(3.1) \quad \frac{e^p}{p^p} \left(\sum_{i=1}^n x_i \right)^p \leq \exp \left(\sum_{i=1}^n x_i \right)$$

and by (2.2) we obtain (1.6). □

Proof of Theorem 1.2. For $(x_1, x_2, \dots, x_n) \in [0, \infty)^n$ and $0 < p \leq 1$, we put $x = \sum_{i=1}^n x_i$. Then by (2.1), we have

$$(3.2) \quad \frac{e^p}{p^p} \left(\sum_{i=1}^n x_i \right)^p \leq \exp \left(\sum_{i=1}^n x_i \right)$$

and by (2.3) we obtain (1.7). □

Proof of Theorem 1.3. This can be concluded by letting $n \rightarrow +\infty$ in Theorem 1.1. □

Proof of Theorem 1.4. By the condition of Theorem 1.4, we have $0 < x_i \leq p$ for all $1 \leq i \leq n$. Then, $x_i^{-p} \geq p^{-p}$ for all $1 \leq i \leq n$. It follows that $\sum_{i=1}^n x_i^{-p} \geq np^{-p}$. Then we obtain

$$(3.3) \quad \sum_{i=1}^n x_i - \ln \left(\sum_{i=1}^n x_i^{-p} \right) \leq np - \ln (np^{-p}) = np + \ln \frac{1}{n} + p \ln p.$$

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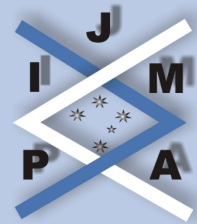
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It follows that

$$\exp\left(\sum_{i=1}^n x_i\right) \leq \frac{p^p}{n} e^{np} \sum_{i=1}^n x_i^{-p}.$$

The proof of Theorem 1.4 is completed. □

Proof of Theorem 1.5. Let $0 < p \leq 1$. By Hölder's inequality, we have

$$(3.4) \quad \int_a^b |f(x)|^p dx \leq \left(\int_a^b |f(x)| dx\right)^p (b-a)^{1-p}.$$

It follows that

$$(3.5) \quad (b-a)^{p-1} \int_a^b |f(x)|^p dx \leq \left(\int_a^b |f(x)| dx\right)^p.$$

On the other hand, by Lemma 2.1, we have

$$(3.6) \quad \frac{e^p}{p^p} \left(\int_a^b |f(x)| dx\right)^p \leq \exp\left(\int_a^b |f(x)| dx\right).$$

By (3.5) and (3.6), we get (1.17). □

Proof of Theorem 1.6. Let $x > 0$ and $t > 0$. Then by Lemma 2.1, we have

$$(3.7) \quad e^t \geq \frac{e^x}{x^x} t^x.$$

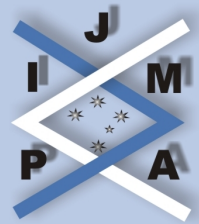
So,

$$(3.8) \quad e^{-t} \geq \frac{e^x}{x^x} t^x e^{-2t}.$$

It is clear that

$$(3.9) \quad 1 \geq \frac{e^x}{x^x} \int_0^\infty t^x e^{-2t} dt = \frac{e^x}{2^{x+1} x^{x-1}} \Gamma(x).$$

The proof of Theorem 1.6 is completed. □



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4. Open Problem

Problem 4.1. For $p \geq 1$ a real number, determine the best possible constant $\alpha \in \mathbb{R}$ such that

$$\frac{e^p}{p^p} \alpha \int_a^b |f(x)|^p dx \leq \exp \left(\int_a^b |f(x)| dx \right).$$



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