

# Journal of Inequalities in Pure and Applied Mathematics

## MULTIVARIATE VERSION OF A JENSEN-TYPE INEQUALITY

ROBERT A. AGNEW

Deerfield, IL 60015-3007, USA.

*EMail:* [raagnew@aol.com](mailto:raagnew@aol.com)



---

volume 6, issue 4, article 120,  
2005.

*Received 12 May, 2005;  
accepted 22 September, 2005.*

*Communicated by: C.E.M. Pearce*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

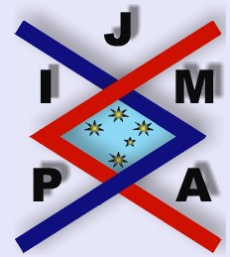
A univariate Jensen-type inequality is generalized to a multivariate setting.

*2000 Mathematics Subject Classification:* Primary 26D15.

*Key words:* Convex functions, Tchebycheff methods, Jensen's inequality.

## Contents

1	Introduction .....	3
2	Main Result .....	4
3	Examples .....	6
	References	



---

### Multivariate Version of a Jensen-Type Inequality

Robert A. Agnew

---

Title Page

Contents



Go Back

Close

Quit

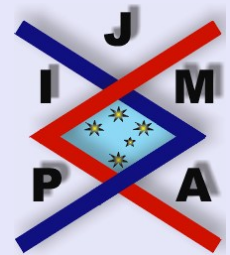
Page 2 of 8

# 1. Introduction

The following theorem was proved in [1], using Tchebycheff methods [4], [5], to extend a result obtained in [2] for the Laplace transform. It was later reproved in [3], [6], [7] using Jensen's inequality.

**Theorem 1.1.** *Let  $X$  be a nonnegative random variable with  $E(X) = \mu > 0$  and  $E(X^2) = \lambda < \infty$ . Suppose that  $f : [0, \infty) \rightarrow \mathbb{R}$  with  $f(0) = 0$  and  $g(x) = f(x)/x$  convex on  $(0, \infty)$ . Then,  $E(f(X)) \geq \mu g(\lambda/\mu) = (\mu^2/\lambda) f(\lambda/\mu)$  and the bound is sharp.*

We next provide a natural multivariate generalization of Theorem 1.1, using the same approach as [1], followed by examples to illustrate its application.



---

Multivariate Version of a  
Jensen-Type Inequality

Robert A. Agnew

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 8

## 2. Main Result

Let  $S = (0, \infty)^n$  and let  $g_1, \dots, g_n$  be real-valued functions on  $S$ . For any column vector  $x = (x_1, \dots, x_n)^T \in S$ , let  $f(x) = \sum_{i=1}^n x_i g_i(x)$  and let  $e_i$  denote the  $i^{\text{th}}$  unit column vector in  $\mathbb{R}^n$ .

**Theorem 2.1.** *Let  $g_1, \dots, g_n$  be convex on  $S$ , and let  $X = (X_1, \dots, X_n)^T$  be a random column vector in  $S$  with  $E(X) = \mu = (\mu_1, \dots, \mu_n)^T$  and  $E(XX^T) = \Sigma + \mu\mu^T$  for covariance matrix  $\Sigma$ . Then,*

$$(2.1) \quad E(f(X)) \geq \sum_{i=1}^n \mu_i g_i \left( \frac{\sum e_i}{\mu_i} + \mu \right)$$

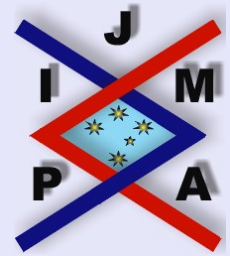
and the bound is sharp.

*Proof.* By convexity, for any  $\xi_i \in S$ , there exists a  $b_i(\xi_i) \in \mathbb{R}^n$  such that

$$(2.2) \quad g_i(x) \geq g_i(\xi_i) + b_i(\xi_i)^T (x - \xi_i)$$

for all  $x \in S$ , i.e., there exists a supporting hyperplane at  $\xi_i$ . Hence,

$$(2.3) \quad \begin{aligned} E(f(X)) &= \sum_{i=1}^n E(X_i g_i(X)) \\ &\geq \sum_{i=1}^n E \left( X_i \left( g_i(\xi_i) + b_i(\xi_i)^T (X - \xi_i) \right) \right) \\ &\geq \sum_{i=1}^n \mu_i \left( g_i(\xi_i) + b_i(\xi_i)^T \left( E \left( \frac{XX_i}{\mu_i} \right) - \xi_i \right) \right) \end{aligned}$$



Multivariate Version of a  
Jensen-Type Inequality

Robert A. Agnew

Title Page

Contents



Go Back

Close

Quit

Page 4 of 8

But

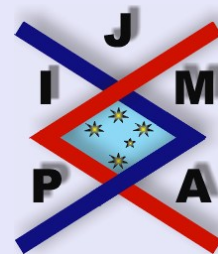
$$E(XX_i) = E(XX^T e_i) = E(XX^T) e_i = \Sigma e_i + \mu \mu_i.$$

Then, (2.2) and (2.3) together imply that

$$\xi_i = E\left(\frac{XX_i}{\mu_i}\right) = \frac{\Sigma e_i}{\mu_i} + \mu$$

yields the maximum bound which is obviously attained when  $X$  is concentrated at  $\mu$ .  $\square$

Theorem 2.1 is a true multivariate extension as the following examples illustrate. As indicated in [2] for the Laplace transform, certain extensions are only nominally multivariate and fall within the domain of Theorem 1.1 because the random variables are combined in a univariate linear combination.



---

### Multivariate Version of a Jensen-Type Inequality

Robert A. Agnew

---

Title Page

Contents



Go Back

Close

Quit

Page 5 of 8

### 3. Examples

**Example 3.1.** Let  $g_i(x) = \alpha_i + \beta_i^T x$  be linear with  $\alpha_i \in \mathbb{R}$  and  $\beta_i \in \mathbb{R}^n$ . Then

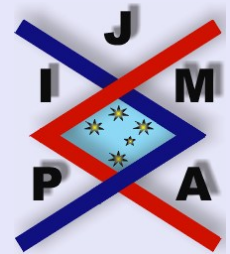
$$f(x) = \sum_{i=1}^n x_i g_i(x) = \sum_{i=1}^n x_i (\alpha_i + \beta_i^T x)$$

is a general quadratic function which can also be written as  $f(x) = \alpha^T x + x^T B x$  where  $\alpha = (\alpha_1, \dots, \alpha_n)^T$  and  $B = [\beta_1, \dots, \beta_n]^T$ . Then we have

$$\begin{aligned} E(f(X)) &= E\left(\sum_{i=1}^n X_i (\alpha_i + \beta_i^T X)\right) \\ &= \sum_{i=1}^n (\alpha_i \mu_i + \beta_i^T (\sum e_i + \mu \mu_i)) \\ &= \sum_{i=1}^n \mu_i \left(\alpha_i + \beta_i^T \left(\frac{\sum e_i}{\mu_i} + \mu\right)\right) \\ &= \alpha^T \mu + \mu^T B \mu + \text{tr}(B \Sigma) \end{aligned}$$

so the Theorem 2.1 bound is, not surprisingly, exact in this general quadratic case.

**Example 3.2.** Let  $g_i(x) = \rho_i \prod_{j=1}^n x_j^{-\gamma_{ij}}$  with  $\rho_i > 0$  and  $\gamma_{ij} > 0$ . Here, the  $g_i$  might represent Cournot-type price functions (inverse demand functions) for quasi-substitutable products where  $x_i$  is the supply of product  $i$  and  $g_i(x_1, \dots, x_n)$  is the equilibrium price of product  $i$ , given its supply and the supplies of its alternates. Then,  $x_i g_i(x)$  represents the revenue from product  $i$  and  $f(x) =$



Multivariate Version of a  
Jensen-Type Inequality

Robert A. Agnew

Title Page

Contents



Go Back

Close

Quit

Page 6 of 8

$\sum_{i=1}^n x_i g_i(x)$  represents total market revenue for the ensemble of products. In this context, we would normally expect  $\gamma_{ij} \in (0, 1)$  for viable products. Then, with probabilistic supplies, we have

$$E(f(X)) \geq \sum_{i=1}^n \mu_i g_i \left( \frac{\sum e_i}{\mu_i} + \mu \right) = \sum_{i=1}^n \mu_i \rho_i \prod_{j=1}^n \left( \frac{\sigma_{ij}}{\mu_i} + \mu_j \right)^{-\gamma_{ij}}$$

where  $\sigma_{ij}$  is the  $ij^{\text{th}}$  element of  $\Sigma$ . This example demonstrates that Theorem 2.1 has an interesting application in economic oligopoly theory.

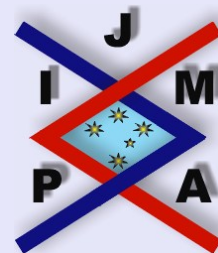
In Example 3.2,  $g_i(x) = e^{h_i(x)}$  where

$$h_i(x) = \ln \rho_i - \sum_{j=1}^n \gamma_{ij} \ln x_j$$

is convex on  $S$ . In general, if  $k : \mathbb{R} \rightarrow \mathbb{R}$  is convex nondecreasing and  $h : S \rightarrow \mathbb{R}$  is convex, then  $g(x) = k(h(x))$  is convex on  $S$  since

$$\begin{aligned} k(h(\lambda x^{(1)} + (1-\lambda)x^{(2)})) &\leq k(\lambda h(x^{(1)}) + (1-\lambda)h(x^{(2)})) \\ &\leq \lambda k(h(x^{(1)})) + (1-\lambda)k(h(x^{(2)})) \end{aligned}$$

for any  $x^{(1)}, x^{(2)} \in S$  and  $\lambda \in [0, 1]$ . Other examples satisfying Theorem 2.1 can be generated by composing the linear functions of Example 3.1 with convex nondecreasing functions like  $k(u) = e^u$ ,  $k(u) = u + \sqrt{u^2 + 1} = e^{\sinh^{-1} u}$ , or  $k(u) = \max(0, u)$ .



Multivariate Version of a  
Jensen-Type Inequality

Robert A. Agnew

Title Page

Contents



Go Back

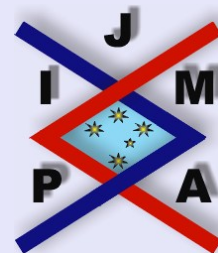
Close

Quit

Page 7 of 8

## References

- [1] R.A. AGNEW, Inequalities with application in economic risk analysis, *J. Appl. Prob.*, **9** (1972), 441–444.
- [2] D. BROOK, Bounds for moment generating functions and for extinction probabilities, *J. Appl. Prob.*, **3** (1966), 171–178.
- [3] B. GULJAŠ, C.E.M. PEARCE AND J. PEČARIĆ, Jensen's inequality for distributions possessing higher moments, with applications to sharp bounds for Laplace-Stieltjes transforms, *J. Austral. Math. Soc. Ser. B*, **40** (1998), 80–85.
- [4] S. KARLIN AND W.J. STUDDEN, *Tchebycheff Systems: with Applications in Analysis and Statistics*, Wiley Interscience, 1966.
- [5] J.F.C. KINGMAN, On inequalities of the Tchebychev type, *Proc. Camb. Phil. Soc.*, **59** (1963), 135–146.
- [6] C.E.M. PEARCE AND J.E. PEČARIĆ, An integral inequality for convex functions, with application to teletraffic congestion problems, *Math. of Opns. Res.*, **20** (1995), 526–528.
- [7] A.O. PITTENGER, Sharp mean-variance bounds for Jensen-type inequalities, *Stat. & Prob. Letters*, **10** (1990), 91–94.



---

### Multivariate Version of a Jensen-Type Inequality

Robert A. Agnew

---

Title Page

Contents



Go Back

Close

Quit

Page 8 of 8