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MULTIVARIATE VERSION OF A JENSEN-TYPE INEQUALITY

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Abstract

A univariate Jensen-type inequality is generalized to a multivariate setting.

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Contents

1. Introduction

The following theorem was proved in [1], using Tchebycheff methods [4], [5], to extend a result obtained in [2] for the Laplace transform. It was later reproved in [3], [6], [7] using Jensen's inequality.

Theorem 1.1. *Let X be a nonnegative random variable with $E(X) = \mu > 0$ and $E(X^2) = \lambda < \infty$. Suppose that $f : [0, \infty) \to \mathbb{R}$ with $f(0) = 0$ and $g(x) = f(x)/x$ convex on $(0, \infty)$. Then, $E(f(X)) \geq \mu g(\lambda/\mu) = (\mu^2/\lambda) f(\lambda/\mu)$ and the bound is sharp.*

We next provide a natural multivariate generalization of Theorem 1.1, using the same approach as [1], followed by examples to illustrate its application.

2. Main Result

Let $S = (0, \infty)^n$ and let $g_1, ..., g_n$ be real-valued functions on S. For any column vector $x = (x_1, \ldots, x_n)^T \in S$, let $f(x) = \sum_{i=1}^{n} x_i g_i(x)$ and let e_i denote the i^{th} unit column vector in $\mathbb{R}^n$.

Theorem 2.1. *Let $g_1, ..., g_n$ be convex on S, and let $X = (X_1, \ldots, X_n)^T$ be a random column vector in S with $E(X) = \mu = (\mu_1, \ldots, \mu_n)^T$ and $E(XX^T) = \Sigma + \mu\mu^T$ for covariance matrix Σ. Then,*

$$E(f(X)) \geq \sum_{i=1}^{n} \mu_i\, g_i\left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right) \tag{2.1}$$

and the bound is sharp.

Proof. By convexity, for any $\xi_i \in S$, there exists a $b_i(\xi_i) \in \mathbb{R}^n$ such that

$$g_i(x) \geq g_i(\xi_i) + b_i(\xi_i)^T (x - \xi_i) \tag{2.2}$$

for all $x \in S$, i.e., there exists a supporting hyperplane at ξ_i. Hence,

$$\begin{aligned} E(f(X)) &= \sum_{i=1}^{n} E(X_i g_i(X)) \\ &\geq \sum_{i=1}^{n} E\left(X_i \left(g_i(\xi_i) + b_i(\xi_i)^T (X - \xi_i)\right)\right) \\ &\geq \sum_{i=1}^{n} \mu_i \left(g_i(\xi_i) + b_i(\xi_i)^T \left(E\left(\frac{XX_i}{\mu_i}\right) - \xi_i\right)\right) \end{aligned} \tag{2.3}$$

But

$$E(XX_i) = E\left(XX^T e_i\right) = E\left(XX^T\right) e_i = \Sigma e_i + \mu\mu_i.$$

Then, (2.2) and (2.3) together imply that

$$\xi_i = E\left(\frac{XX_i}{\mu_i}\right) = \frac{\Sigma e_i}{\mu_i} + \mu$$

yields the maximum bound which is obviously attained when X is concentrated at μ. □

Theorem 2.1 is a true multivariate extension as the following examples illustrate. As indicated in [2] for the Laplace transform, certain extensions are only nominally multivariate and fall within the domain of Theorem 1.1 because the random variables are combined in a univariate linear combination.

3. Examples

Example 3.1. *Let* $g_i(x) = \alpha_i + \beta_i^T x$ *be linear with* $\alpha_i \in \mathbb{R}$ *and* $\beta_i \in \mathbb{R}^n$ *. Then*

$$f(x) = \sum_{i=1}^{n} x_i\, g_i(x) = \sum_{i=1}^{n} x_i \left(\alpha_i + \beta_i^T x\right)$$

is a general quadratic function which can also be written as $f(x) = \alpha^T x + x^T B x$ *where* $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ *and* $B = [\beta_1, \ldots, \beta_n]^T$*. Then we have*

$$\begin{aligned} E(f(X)) &= E\left(\sum_{i=1}^{n} X_i \left(\alpha_i + \beta_i^T X\right)\right) \\ &= \sum_{i=1}^{n} \left(\alpha_i\, \mu_i + \beta_i^T \left(\Sigma\, e_i + \mu \mu_i\right)\right) \\ &= \sum_{i=1}^{n} \mu_i \left(\alpha_i + \beta_i^T \left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right)\right) \\ &= \alpha^T \mu + \mu^T B\, \mu + tr\, (B\Sigma) \end{aligned}$$

so the Theorem 2.1 bound is, not surprisingly, exact in this general quadratic case.

Example 3.2. *Let* $g_i(x) = \rho_i \prod_{j=1}^{n} x_j^{-\gamma_{ij}}$ *with* $\rho_i > 0$ *and* $\gamma_{ij} > 0$*. Here, the* g_i *might represent Cournot-type price functions (inverse demand functions) for quasi-substitutable products where* x_i *is the supply of product* i *and* $g_i(x_1, \ldots, x_n)$ *is the equilibrium price of product* i*, given its supply and the supplies of its alternates. Then,* $x_i\, g_i(x)$ *represents the revenue from product* i *and* $f(x) =$

$\sum_{i=1}^{n} x_i\, g_i(x)$ represents total market revenue for the ensemble of products. In this context, we would normally expect $\gamma_{ij} \in (0,1)$ for viable products. Then, with probabilistic supplies, we have

$$E(f(X)) \geq \sum_{i=1}^{n} \mu_i\, g_i\left(\frac{\Sigma\, e_i}{\mu_i} + \mu\right) = \sum_{i=1}^{n} \mu_i\, \rho_i \prod_{j=1}^{n} \left(\frac{\sigma_{ij}}{\mu_i} + \mu_j\right)^{-\gamma_{ij}}$$

where σ_{ij} is the ij^{th} element of Σ. This example demonstrates that Theorem 2.1 has an interesting application in economic oligopoly theory.

In Example 3.2, $g_i(x) = e^{h_i(x)}$ where

$$h_i(x) = ln\, \rho_i - \sum_{j=1}^{n} \gamma_{ij} \ln x_j$$

is convex on S . In general, if $k : \mathbb{R} \to \mathbb{R}$ is convex nondecreasing and $h : S \to \mathbb{R}$ is convex, then $g(x) = k(h(x))$ is convex on S since

$$\begin{aligned} k\left(h\left(\lambda\, x^{(1)} + (1-\lambda)\, x^{(2)}\right)\right) &\leq k\left(\lambda\, h\left(x^{(1)}\right) + (1-\lambda)\, h\left(x^{(2)}\right)\right) \\ &\leq \lambda\, k\left(h\left(x^{(1)}\right)\right) + (1-\lambda)\, k\left(h\left(x^{(2)}\right)\right) \end{aligned}$$

for any $x^{(1)}, x^{(2)} \in S$ and $\lambda \in [0,1]$. Other examples satisfying Theorem 2.1 can be generated by composing the linear functions of Example 3.1 with convex nondecreasing functions like $k(u) = e^u$, $k(u) = u + \sqrt{u^2+1} = e^{\sinh^{-1} u}$, or $k(u) = \max(0,u)$.

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