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UNIQUENESS OF A MEROMORPHIC FUNCTION AND ITS DERIVATIVE

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ABSTRACT. In the paper we consider the problem of uniqueness of meromorphic functions sharing one finite nonzero value or one finite nonzero function with their derivatives and answer some open questions posed by K.W. Yu.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

Let f, g be nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicity and we say that f, g share the value a IM (ignoring multiplicities) if f, g have the same a-points and the multiplicities are not taken into account.

We do not explain the standard notations of the value distribution theory as these are available in [3]. However in the following definition we explain some notations used in the paper.

Definition 1.1. For two meromorphic functions f, g and for $a, b \in \mathbb{C} \cup \{\infty\}$ and for a positive integer k

(*i*) $N(r, a; f \mid \geq k)$ $(\overline{N}(r, a; f \mid \geq k))$ denotes the counting function (reduced countion function) of those a-points of f whose multiplicities are not less than k,

(*ii*) $N(r, a; f \mid g = b)$ $(\overline{N}(r, a; f \mid g = b))$ denotes the counting function (reduced counting function) of those a-points of f which are the b-points of g,

(*iii*) $N(r, a; f \mid g \neq b)$ $(\overline{N}(r, a; f \mid g \neq b))$ denotes the counting function (reduced counting function) of those a-points of f which are not the b-points of g,

(*iv*) $N_p(r, a; f) = \overline{N}(r, a; f) + \sum_{k=2}^{p} \overline{N}(r, a; f \mid \geq k)$,

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(v) $N_2(r, a; f \mid g = b)$ $(N_2(r, a; f \mid g \neq b))$ denotes the counting function of those a-points of f which are (are not) the b-points of g, where an a-point of f with multiplicity m is counted m times if $m \leq 2$ and twice if $m > 2$,

(vi) $N(r, a; f \mid\leq k)$ $(\overline{N}(r, a; f \mid\leq k))$ denotes the counting function (reduced countion function) of those a-points of f whose multiplicities are not greater than k.

Definition 1.2. Let f and g share a value a IM. Let z be an a-point of f and g with multiplicities $p_f(z)$ and $p_g(z)$ respectively. We put

$$\begin{aligned}\overline{\nu}_f(z) &= 1 \text{ if } p_f(z) \neq p_g(z)\\ &= 0 \text{ if } p_f(z) = p_g(z).\end{aligned}$$

Let $\overline{n}_*(r, a; f, g) = \sum_{|z|\leq r} \overline{\nu}_f(z)$ and $\overline{N}_*(r, a; f, g)$ be the integrated counting function obtained from $\overline{n}_*(r, a; f, g)$ in the usual manner.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$.

Rubel-Yang [8], Mues-Steinmetz [7], Gundersen [2], Yang [9] considered the uniqueness problem of entire functions with their first and k^{th} derivatives involving two CM or IM values.

R. Brück [1] considered the uniqueness problem of an entire function when it shares a single value CM with its derivative and proved the following theorem.

Theorem A. [1] *Let f be a nonconstant entire function. If f and f' share the value 1 CM and $N(r, 0; f') = S(r, f)$ then $\frac{f'-1}{f-1}$ is a nonzero constant.*

For entire functions of finite order Yang [10] improved Theorem A and proved the following result.

Theorem B. [10] *Let f be a nonconstant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where $k(\geq 1)$ is an integer.*

Zhang [12] extended Theorem A to meromorphic functions and proved the following results.

Theorem C. [12] *Let f be a nonconstant meromorphic function. If f and f' share 1 CM and if*

$$\overline{N}(r, \infty; f) + N(r, 0; f') < \{\lambda + o(1)\}T(r, f') \tag{1.1}$$

for some constant $\lambda \in (0; 1/2)$, then $\frac{f'-1}{f-1}$ is a nonzero constant.

Theorem D. [12] *Let f be a nonconstant meromorphic function. If f and $f^{(k)}$ share 1 CM and if*

$$2\overline{N}(r, \infty; f) + \overline{N}(r, 0; f') + N(r, 0; f^{(k)}) < \{\lambda + o(1)\}T(r, f^{(k)}) \tag{1.2}$$

for some constant $\lambda \in (0; 1)$, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.

Considering $f(z) = 1 + \tan z$ we can verify that in Theorems C and D it is not possible to relax simultaneously the conditions (1.1) and (1.2) respectively and the nature of sharing the value from CM to IM. Naturally one will desire to see how far it is possible to relax the nature of sharing the value 1. In the paper we deal with this problem with the aid of the notion of weighted sharing of values as introduced in [4, 5] and we see that it is indeed possible to some extent, at the cost of some change in the condition (1.2).

Zheng-Wang [13] considered the uniqueness problem of entire functions sharing two small functions CM with their derivatives. Recently Yu [11] considered the uniqueness problem of an entire or meromorphic function when it shares one small function with its derivative. He proved the following two theorems.

Theorem E. [11] *Let f be a nonconstant entire function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0;f) > 3/4$ then $f \equiv f^{(k)}$, where k is a positive integer.*

Theorem F. [11] *Let f be a nonconstant nonentire meromorphic function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r,a) = o\{T(r,f)\}$ as $r \to \infty$. If*

(*i*) *f and a have no common pole,*
(*ii*) *$f - a$ and $f^{(k)} - a$ share the value 0 CM,*
(*iii*) *$4\delta(0,f) + 2(8+k)\Theta(\infty;f) > 19 + 2k$,*

then $f \equiv f^{(k)}$, where k is a positive integer.

Yu [11] further showed that the condition (i) of Theorem F can be dropped if k is an odd integer. In the same paper Yu [11] posed the following open questions:

(1) Can CM shared value be replaced by an IM shared value ?
(2) Can the condition $\delta(0;f) > 3/4$ of Theorem E be further relaxed ?
(3) Can the condition(iii) of Theorem F be further relaxed ?
(4) Can, in general, the condition (i) of Theorem F be dropped ?

Although the fourth question is still open, in the paper we give some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of a. In the following definition we explain the idea of weighted sharing of values which measures how close a shared value is to be shared IM or to be shared CM.

Definition 1.3. [4, 5] Let k be a nonnegative integer or infinity . For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_o is an a-point of f with multiplicity $m(\le k)$ if and only if it is an a-point of g with multiplicity $m(\le k)$ and z_o is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.4. We denote by $\delta_p(a;f)$ the quantity

$$\delta_p(a;f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)},$$

where p is a positive integer.

Clearly $\delta_p(a;f) \ge \delta(a;f)$.

We now state the main results of the paper.

Theorem 1.1. *Let f be a nonconstant meromorphic function and k be a positive integer. If f, $f^{(k)}$ share $(1,2)$ and*

$$2\overline{N}(r,\infty;f) + N_2(r,0;f^{(k)}) + N_2(r,0;f') < \{\lambda + o(1)\}T(r,f^{(k)}) \tag{1.3}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.

The following corollary follows from Theorem 1.1 for $k = 1$ and improves Theorem C.

Corollary 1.2. *Theorem C holds if the condition (1.1) is replaced by the following*

$$\overline{N}(r,\infty;f)+N_2(r,0;f')<\{\lambda+o(1)\}T(r,f')$$

for some constant $\lambda\in(0,1/2)$.

Theorem 1.3. *Let f be a nonconstant meromorphic function and k be a positive integer. If f, $f^{(k)}$ share $(1,1)$ and*

$$2\overline{N}(r,\infty;f)+N_2(r,0;f^{(k)})+2\overline{N}(r,0;f')<\{\lambda+o(1)\}T(r,f^{(k)}) \tag{1.4}$$

for $r\in I$, where $0<\lambda<1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.

If f, $f^{(k)}$ share $(1,0)$, it is clear that f does not possess any 1-point with multiplicity greater than k. So if in Theorem 1.1 and in Theorem 1.3 we respectively put $k\le 2$ and $k=1$, it follows that f, $f^{(k)}$ practically share $(1,\infty)$. It then follows from the proof that in these cases we can replace each of the conditions (1.3) and (1.4) by the following

$$2\overline{N}(r,\infty;f)+N_2(r,0;f^{(k)})+\overline{N}(r,0;f')<\{\lambda+o(1)\}T(r,f^{(k)})$$

for $r\in I$, where $0<\lambda<1$ and I is a set of infinite linear measure.

It is clear that if f and $f^{(k)}$ satisfy the conclusions of Theorems 1.1, 1.3 then $f=Ae^{\mu z}+1-1/c$, where A,c are nonzero constants and μ is a k^{th} root of c. So it follows that the conditions of the theorems are necessary.

Theorem 1.4. *Let f be a nonconstant meromorphic function and k be a positive integer. Let $a\equiv a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a)=S(r,f)$. If*

(i) *a has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,*
(ii) *$f-a$ and $f^{(k)}-a$ share $(0,2)$,*
(iii) *$2\delta_{2+k}(0;f)+(4+k)\Theta(\infty;f)>5+k$, then $f\equiv f^{(k)}$.*

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [3, p. 55]. *Let f be a nonconstant meromorphic function. Then*

$$T(r,f^{(k)})\le(1+k)T(r,f)+S(r,f).$$

Lemma 2.2. *If f is a nonconstant meromorphic function and f, $f^{(k)}$ share $(1,0)$ then*

$$T(r,f)\le\left(k+2+\frac{1}{1+k}\right)T(r,f^{(k)})+S(r,f),$$

where k is a positive integer.

Proof. By Milloux's basic result [3, p. 57] we get

$$T(r,f)\le\overline{N}(r,\infty;f)+N(r,0;f)+\overline{N}(r,1;f^{(k)})-N_0(r,0;f^{(1+k)})+S(r,f),$$

where $N_0(r,0;f^{(1+k)})$ is the counting function of those zeros of $f^{(1+k)}$ which are not the zeros of $f^{(k)}-1$.

Since

$$N(r,0;f)-N_0(r,0;f^{(1+k)})\le(1+k)\overline{N}(r,0;f)$$

and

$$(1+k)\overline{N}(r,\infty;f)\le N(r,\infty;f^{(k)})\le T(r,f^{(k)}),$$

it follows that

$$T(r,f) \le \frac{1}{1+k} T(r,f^{(k)}) + (1+k)\overline{N}(r,0;f) + \overline{N}(r,1;f^{(k)}) + S(r,f).$$

Applying this inequality to $f-1$ and noting that f, $f^{(k)}$ share $(1,0)$ we obtain

$$\begin{aligned} T(r,f) &\le \frac{1}{1+k} T(r,f^{(k)}) + (1+k)\overline{N}(r,1;f) + \overline{N}(r,1;f^{(k)}) + S(r,f) \\ &\le \left(2+k+\frac{1}{1+k}\right) T(r,f^{(k)}) + S(r,f). \end{aligned}$$

This proves the lemma. □

Lemma 2.3. *Let f be a nonconstant meromorphic function and k be a positive integer. Then*

$$N_2(r,0;f^{(k)}) \le k\overline{N}(r,\infty;f) + N_{2+k}(r,0;f) + S(r,f).$$

Proof. By the first fundamental theorem and the Milloux theorem [3, p. 55] we get

$$\begin{aligned} N\left(r,0;f^{(1)} \mid f \ne 0\right) &= N\left(r,0;\frac{f^{(1)}}{f}\right) \\ &\le N\left(r,\infty;\frac{f^{(1)}}{f}\right) + S(r,f) \\ &= \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + S(r,f). \end{aligned}$$

Also for a positive integer p

$$N_p(r,0;f^{(1)} \mid f=0) = N(r,0;f \mid\le p) - \overline{N}(r,0;f \mid\le p) + p\overline{N}(r,0;f \mid\ge 1+p).$$

So we get

$$\begin{aligned} N_p(r,0;f^{(1)}) &\le N(r,0;f^{(1)} \mid f \ne 0) + N_p(r,0;f^{(1)} \mid f=0) \\ &\le \overline{N}(r,\infty;f) + N_{p+1}(r,0;f) + S(r,f). \end{aligned} \tag{2.1}$$

For $p=2$ we get from (2.1)

$$N_2(r,0;f^{(1)}) \le \overline{N}(r,\infty;f) + N_{2+1}(r,0;f) + S(r,f),$$

which is the lemma for $k=1$.

Suppose that the lemma is true for $k=m$. Then in view of (2.1) for $p=2+m$ and Lemma 2.1 we get

$$\begin{aligned} N_2(r,0;f^{(m+1)}) &= N_2(r,0;\left(f^{(1)}\right)^{(m)}) \\ &\le m\overline{N}(r,\infty;f^{(1)}) + N_{2+m}(r,0;f^{(1)}) + S(r,f^{(1)}) \\ &\le (m+1)\overline{N}(r,\infty;f) + N_{2+(m+1)}(r,0;f) + S(r,f), \end{aligned}$$

which is the lemma for $k=m+1$. So by mathematical induction the lemma is proved. □

Lemma 2.4. [5] *Let f and g be two meromorphic functions sharing $(1,2)$. Then one of the following holds:*

(*i*) $T(r) \le N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r,f)+S(r,g)$, *where* $T(r)=\max\{T(r,f),T(r,g)\}$*;*
(*ii*) $fg \equiv 1$*;*
(*iii*) $f \equiv g$.

Lemma 2.5. [6] *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,\alpha)=S(r,f)$. Suppose that b and c are any two finite nonzero distinct complex numbers. If $\psi=\alpha\,(f)^n\left(f^{(k)}\right)^p$, where $n(\geq 0)$, $p(\geq 1)$ and $k(\geq 1)$ are integers, then*

$$(p+n)T(r,f)\leq (p+n)N(r,0;f)+N(r,b;\psi)+N(r,c;\psi) - N(r,\infty;f)-N(r,0;\psi')+S(r,f).$$

Lemma 2.6. *Let f be a nonconstant meromorphic function and k be a positive integer. If f, $f^{(k)}$ share $(1,0)$ and $f^{(k)}=\frac{Af+B}{Cf+D}$, where A,B,C,D are constants, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.*

Proof. Since f is nonconstant and f, $f^{(k)}$ share $(1,0)$, $f^{(k)}$ is also nonconstant and so $AD-BC\neq 0$. If z_0 is a pole of f with multiplicity p then z_0 is either a regular point or a pole with multiplicity p of $\frac{Af+B}{Cf+D}$ but z_0 is a pole of $f^{(k)}$ with multiplicity $p+k$. So f and $f^{(k)}$ have no pole.

Now we consider the following cases.

Case 1. Let $C\neq 0$. Since $f^{(k)}$ has no pole, it follows that $f+D/C$ has no zero. Differentiating $f^{(k)}=\frac{Af+B}{Cf+D}$ we get

$$\frac{f^{(1+k)}}{f^{(1)}}=\frac{AD-BC}{(Cf+D)^2}.$$

This shows that $\frac{f^{(1+k)}}{f^{(1)}}$ has no zero and pole. Now in view of Lemma 2.1 we get

$$\begin{aligned}\overline{N}(r,1;f^{(k)})=\overline{N}(r,1;f)&\leq \overline{N}\left(r,\frac{AD-BC}{(C+D)^2};\frac{f^{(1+k)}}{f^{(1)}}\right)\\ &\leq T\left(r,\frac{f^{(1+k)}}{f^{(1)}}\right)=m\left(r,\frac{f^{(1+k)}}{f^{(1)}}\right)=S(r,f).\end{aligned}$$

Hence in view of the second fundamental theorem we get $D+C=0$. So $f-1$ has no pole and no zero and we can put $f-1=\exp(g)$, where g is an entire function. Since $f^{(1)}=g^{(1)}\exp(g)$, it follows that $N(r,0;f^{(1)})=N(r,0;g^{(1)})=S(r,\exp(g))=S(r,f)$.

Now we get by Lemmas 2.1, 2.2 and 2.3

$$\begin{aligned}\overline{N}(r,0;f^{(k)})\leq N_2(r,0;f^{(k)})&\leq (k-1)\overline{N}(r,\infty;f^{(1)})+N_{1+k}(r,0;f^{(1)})+S(r,f^{(1)})\\ &\leq N(r,0;f^{(1)})+S(r,f)=S(r,f)=S(r,f^{(k)}),\end{aligned}$$

which implies a contradiction because $f^{(k)}$ has no pole and no 1-point.

Case 2. Let $C=0$. Then clearly $AD\neq 0$ and

$$f^{(k)}=\gamma f+\delta, \tag{2.2}$$

where $\gamma=A/D$ and $\delta=B/D$.

First we suppose that f and so $f^{(k)}$ has no 1-point. If $\gamma+\delta=0$ then $f^{(k)}=\gamma(f-1)$ and so $f^{(k)}$ has no zero. Hence $f^{(k)}$ has no zero, pole and 1-point, which is impossible.

Let $\gamma+\delta\neq 0$. Since f has no pole and no 1-point, it follows from (2.2) that $f^{(k)}$ has no pole, 1-point and $(\gamma+\delta)$-point. So in view of the second fundamental theorem we get $\gamma+\delta=1$ and from (2.2) we see that $f^{(k)}=\gamma f+1-\gamma$.

Finally we suppose that f and $f^{(k)}$ has at least one 1-point. Then from (2.2) we get $\gamma+\delta=1$ and so $f^{(k)}=\gamma f+1-\gamma$. This proves the lemma.

□

Lemma 2.7. [5] *Let f, g be meromorphic functions sharing $(1,1)$ and*

$$h=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right).$$

Then $N(r,1;f\mid\le 1)=N(r,1;g\mid\le 1)\le N(r,h)+S(r,f)+S(r,g)$.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.4. Let $\phi=f/a$ and $\psi=f^{(k)}/a$. Then ϕ and ψ share $(1,2)$. If possible, suppose that

$$T(r,\phi)\le N_2(r,0;\phi)+N_2(r,0;\psi)+N_2(r,\infty;\phi)+N_2(r,\infty;\psi)+S(r,\phi)+S(r,\psi).$$

Then it follows in view of Lemmas 2.1 and 2.3 that

$$\begin{aligned}T(r,f)&\le N_2(r,0;f)+N_2(r,0;f^{(k)})+4\overline{N}(r,\infty;f)+S(r,f)\\&\le 2N_{2+k}(r,0;f)+(4+k)\overline{N}(r,\infty;f)+S(r,f)\end{aligned}$$

and so

$$2\delta_{2+k}(0;f)+(4+k)\Theta(\infty;f)\le 5+k,$$

a contradiction.

If possible, suppose that $\phi\psi\equiv 1$. So

$$ff^{(k)}\equiv a^2. \tag{3.1}$$

If f is a rational function then a becomes a nonzero constant. So from (3.1) we see that f has no zero and pole. Since f is nonconstant, this is a contradiction.

If f is transcendental then by Lemma 2.5 we get in view of (3.1)

$$\begin{aligned}2T(r,f)&\le 2N(r,0;f)+2T(r,ff^{(k)})+S(r,f)\\&=2N(r,0;f)+S(r,f)\\&\le 2N(r,0;a^2)+S(r,f)\\&=S(r,f),\end{aligned}$$

a contradiction.

Therefore by Lemma 2.4 we get $\phi\equiv\psi$ and so $f\equiv f^{(k)}$. This proves the theorem. □

Proof of Theorem 1.3. Let

$$H=\left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{f^{(2+k)}}{f^{(1+k)}}-\frac{2f^{(1+k)}}{f^{(k)}-1}\right).$$

We denote by $\overline{N}_0(r,0;f^{(1+k)})$ the reduced counting function of those zeros of $f^{(1+k)}$ which are not the zeros of $f'(f^{(k)}-1)f^{(k)}$. Let $H\not\equiv 0$. Since H has only simple poles, it follows that

$$\begin{aligned}N(r,H)\le{}&\overline{N}(r,\infty;f)+\overline{N}_*(r,1;f,f^{(k)})+\overline{N}(r,0;f^{(k)}\mid\ge 2)\\&+\overline{N}(r,0;f')-\overline{N}(r,1;f\mid\ge 2)+\overline{N}_0(r,0;f^{(1+k)}).\end{aligned} \tag{3.2}$$

Now by Lemmas 2.1, 2.2 and 2.7 we get from (3.2) because f, $f^{(k)}$ share $(1,1)$ and so $\overline{N}_*(r,1;f,f^{(k)}) \le \overline{N}(r,1;f\mid\ge 2)$

$$\begin{aligned}\overline{N}(r,1;f^{(k)}) &= \overline{N}(r,1;f)\\ &= \overline{N}(r,1;f\mid\le 1) + \overline{N}(r,1;f\mid\ge 2)\\ &\le N(r,H) + \overline{N}(r,1;f\mid\ge 2) + S(r,f^{(k)})\\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}\mid\ge 2) + \overline{N}(r,0;f')\\ &\qquad + \overline{N}(r,1;f\mid\ge 2) + \overline{N}_0(r,0;f^{(1+k)}) + S(r,f^{(k)})\\ &\le \overline{N}(r,\infty;f) + 2\overline{N}(r,0;f') + \overline{N}(r,0;f^{(k)}\mid\ge 2)\\ &\qquad + \overline{N}_0(r,0;f^{(1+k)}) + S(r,f^{(k)}).\end{aligned} \tag{3.3}$$

By the second fundamental theorem we get in view of (3.3)

$$\begin{aligned}T(r,f^{(k)}) &\le \overline{N}(r,0;f^{(k)}) + \overline{N}(r,1;f^{(k)}) + \overline{N}(r,\infty;f^{(k)}) - \overline{N}(r,0;f^{(1+k)}) + S(r,f^{(k)})\\ &\le 2\overline{N}(r,\infty;f) + 2\overline{N}(r,0;f') + N_2(r,0;f^{(k)}) + S(r,f^{(k)}),\end{aligned}$$

which contradicts the given condition.

Hence $H \equiv 0$ and so $f^{(k)} = \frac{Af+B}{Cf+D}$, where A,B,C,D are constants. Now the theorem follows from Lemma 2.6. □

Proof of Theorem 1.1. Let H be given as in the proof of Theorem 1.3 and $H \not\equiv 0$. Since f, $f^{(k)}$ share $(1,2)$ and so $\overline{N}_*(r,a;f,f^{(k)}) \le \overline{N}(r,1;f\mid\ge 3)$, we get from (3.2) by Lemmas 2.1, 2.2 and 2.7

$$\begin{aligned}\overline{N}(r,1;f^{(k)}) &= \overline{N}(r,1;f)\\ &= \overline{N}(r,1;f\mid\le 1) + \overline{N}(r,1;f\mid\ge 2)\\ &\le N(r,H) + \overline{N}(r,1;f\mid\ge 2) + S(r,f^{(k)})\\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}\mid\ge 2) + \overline{N}(r,0;f')\\ &\qquad + \overline{N}(r,1;f\mid\ge 3) + \overline{N}_0(r,0;f^{(1+k)}) + S(r,f^{(k)})\\ &\le \overline{N}(r,\infty;f) + \overline{N}(r,0;f^{(k)}\mid\ge 2) + \overline{N}(r,0;f')\\ &\qquad + \overline{N}(r,0;f'\mid\ge 2) + \overline{N}_0(r,0;f^{(1+k)}) + S(r,f^{(k)})\\ &= \overline{N}(r,\infty;f) + N_2(r,0;f') + \overline{N}(r,0;f^{(k)})\mid\ge 2)\\ &\qquad + \overline{N}_0(r,0;f^{(1+k)}) + S(r,f^{(k)}).\end{aligned} \tag{3.4}$$

By the second fundamental theorem we get in view of (3.4)

$$\begin{aligned}T(r,f^{(k)}) &\le \overline{N}(r,0;f^{(k)}) + \overline{N}(r,1;f^{(k)}) + \overline{N}(r,\infty;f^{(k)}) - \overline{N}(r,0;f^{(1+k)}) + S(r,f^{(k)})\\ &\le 2\overline{N}(r,\infty;f) + N_2(r,0;f') + N_2(r,0;f^{(k)}) + S(r,f^{(k)}),\end{aligned}$$

which contradicts the given condition.

Hence $H \equiv 0$ and so $f^{(k)} = \frac{Af+B}{Cf+D}$, where A,B,C,D are constants. Now the theorem follows from Lemma 2.6. □

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