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ON EMBEDDING OF THE CLASS H^ω

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ABSTRACT. In [4] we extended an interesting theorem of Medvedeva [5] pertaining to the embedding relation $H^\omega \subset \Lambda BV$, where ΛBV denotes the set of functions of Λ-bounded variation, which is encountered in the theory of Fourier trigonometric series. Now we give a further generalization of our result. Our new theorem tries to unify the notion of φ-variation due to Young [6], and that of the generalized Wiener class $BV(p(n)\uparrow)$ due to Kita and Yoneda [3]. For further references we refer to the paper by Goginava [2].

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1. INTRODUCTION

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0,1]$ having the following properties:

$$\omega(0) = 0,\ \omega(\delta_1+\delta_2) \le \omega(\delta_1)+\omega(\delta_2) \text{ for } 0 \le \delta_1 \le \delta_2 \le \delta_1+\delta_2 \le 1.$$

Such a function is called a modulus of continuity, and it will be denoted by $\omega(\delta) \in \Omega$.

The modulus of continuity of a continuous function f will be denoted by $\omega(f;\delta)$, that is,

$$\omega(f;\delta) := \sup_{\substack{0\le h\le\delta \\ 0\le x\le 1-h}} |f(x+h)-f(x)|.$$

As usual, set

$$H^\omega := \{f \in C : \omega(f;\delta) = O(\omega(\delta))\}.$$

If $\omega(\delta) = \delta^\alpha,\ 0<\alpha\le 1$ we write H^α instead of H^{δ^α}.

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Finally we define a new class of real functions $f : [0,1] \to \mathbb{R}$. For every $k \in \mathbb{N}$ let $\varphi_k : [0,\infty) \to \mathbb{R}$ be a nondecreasing function with $\varphi_k(0) = 0$; and let $\Lambda := \{\lambda_k\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

If a function $f : [0,1] \to \mathbb{R}$ satisfies the condition

$$\sup \sum_{k=1}^{N} \varphi_k(|f(b_k) - f(a_k)|)\lambda_k^{-1} < \infty, \tag{1.1}$$

where the supremum is extended over all systems of nonoverlapping subintervals (a_k, b_k) of $[0,1]$, then f is said to be of $\Lambda\{\varphi_k\}$-bounded variation, and this fact is denoted by $f \in \Lambda\{\varphi_k\}BV$. In the special cases when all $\varphi_k(x) = \varphi(x)$, we write $f \in \Lambda_\varphi BV$ (see [4]), and if $\varphi(x) = x^p$ we use the notation $f \in \Lambda_p BV$, and when $p = 1$, simply $f \in \Lambda BV$ (see [5]). In the case $\lambda_k = 1$ and $\varphi_k(x) = \varphi(x)$ for all k, then we get the class V_φ due to Young [6], finally if $\lambda_k = 1$ and $\varphi_k(x) = x^{p_k}$, $p_k \uparrow$, we get a class similar to $BV(p(n) \uparrow)$ (see [3]).

Medvedeva [5] proved the following useful theorem, among others.

Theorem 1.1. *The embedding relation $H^\omega \subset \Lambda BV$ holds if and only if*

$$\sum_{k=1}^{\infty} \omega(t_k)\lambda_k^{-1} < \infty$$

for any sequence $\{t_k\}$ satisfying the conditions:

$$t_k \geq 0, \qquad \sum_{k=1}^{\infty} t_k \leq 1. \tag{1.2}$$

In the sequel, the fact that a sequence $t := \{t_k\}$ has the properties (1.2) will be denoted by $t \in T$. K and K_i will denote positive constants, not necessarily the same at each occurrence.

Among others, in [4] we showed that if $0 < \alpha \leq 1$ and $p\alpha \geq 1$ then $H^\alpha \subset \Lambda_p BV$ always holds, furthermore that if $0 < p < 1/\alpha$, then $H^\alpha \subset \Lambda_p BV$ is fulfilled if and only if for any $t \in T$,

$$\sum_{k=1}^{\infty} t_k^{\alpha p}\lambda_k^{-1} < \infty.$$

If $\omega(\delta)$ is a general modulus of continuity then for $0 < p < 1$ we verified that $H^\omega \subset \Lambda_p BV$ holds if and only if for any $t \in T$

$$\sum_{k=1}^{\infty} \omega(t_k)^p\lambda_k^{-1} < \infty. \tag{1.3}$$

These latter two results are immediate consequences of the following theorem of [4].

Theorem 1.2. *Assume that $\varphi(x)$ is a function such that $\varphi(\omega(\delta)) \in \Omega$. Then $H^\omega \subset \Lambda_\varphi BV$ holds if and only if for any $t \in T$*

$$\sum_{k=1}^{\infty} \varphi(\omega(t_k))\lambda_k^{-1} < \infty.$$

Remark 1.3. It would be of interest to mention that by Theorem 1.2 the restriction $0 < p < 1$ claimed above, can be replaced by the weaker condition $\omega(\delta)^p \in \Omega$, and then the embedding relation $H^\omega \subset \Lambda_p BV$ also holds if and only if (1.3) is true.

2. RESULTS

Our new theorem tries to unify and generalize all of the former results.

Theorem 2.1. *Assume that $\omega(t) \in \Omega$ and for every $k \in \mathbb{N}$, $\varphi_k(\omega(\delta)) \in \Omega$. Then the embedding relation $H^\omega \subset \Lambda\{\varphi_k\}BV$ holds if and only if for any $t \in T$*

$$\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} < \infty. \tag{2.1}$$

Our theorem plainly yields the following assertion.

Corollary 2.2. *If for all $k \in \mathbb{N}$, $p_k > 0$ and $\omega(\delta)^{p_k} \in \Omega$, that is, if $\varphi_k(x) = x^{p_k}$, then $H^\omega \subset \Lambda\{x^{p_k}\}BV$ holds if and only if for any $t \in T$*

$$\sum_{k=1}^{\infty} \omega(t_k)^{p_k}\lambda_k^{-1} < \infty. \tag{2.2}$$

It is also obvious that if $\omega(\delta) = \delta^\alpha,\ 0 < \alpha \le 1$, then (2.1) and (2.2) reduce to

$$\sum_{k=1}^{\infty} \varphi_k(t_k^\alpha)\lambda_k^{-1} < \infty \text{ and } \sum_{k=1}^{\infty} t_k^{\alpha p_k}\lambda_k^{-1} < \infty,$$

respectively.

3. LEMMAS

In the proof we shall use the following three lemmas.

Lemma 3.1 ([1, p. 78])**.** *If $\omega(\delta) \in \Omega$ then there exists a concave function $\omega^*(\delta)$ such that*

$$\omega(\delta) \le \omega^*(\delta) \le 2\omega(\delta).$$

Lemma 3.2. *If $\omega(\delta) \in \Omega$ and $t = \{t_k\} \in T$, then there exists a function $f \in H^\omega$ such that if*

$$x_0 = 0,\ x_1 = \frac{t_1}{2},$$

$$x_{2n} = \sum_{i=1}^{n} t_i \text{ and } x_{2n+1} = x_{2n} + \frac{t_{n+1}}{2},\ n \ge 1,$$

then

$$f(x_{2n}) = 0 \text{ and } f(x_{2n+1}) = \omega(t_{n+1}) \text{ for all } n \ge 0.$$

A concrete function with these properties is given in [5].

Lemma 3.3. *If $\omega(t) \in \Omega$ and for all $k \in \mathbb{N},\ \varphi_k(\omega(t)) \in \Omega$ also holds, furthermore for any $t \in T$ the condition (2.1) stays, then there exists a positive number M such that for any $t \in T$*

$$\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} \le M \tag{3.1}$$

holds.

Proof of Lemma 3.3. The proof follows the lines given in the proof of Theorem 2 emerging in [5]. Without loss of generality, due to Lemma 3.1, we can assume that, for every k, the functions $\varphi_k(\omega(\delta))$ are concave moduli of continuity.

Indirectly, let us suppose that there is no number M with property (3.1). Then for any $i \in \mathbb{N}$ there exists a sequence $t^{(i)} := \{t_{k,i}\} \in T$ such that

$$2^i < \sum_{k=1}^{\infty} \varphi_k(\omega(t_{k,i}))\lambda_k^{-1} < \infty. \tag{3.2}$$

Now define

$$t_k := \sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i}.$$

It is easy to see that $t := \{t_k\} \in T$, and thus (2.1) also holds.

Since every $\varphi_k(\omega(\omega))$ is concave, thus by Jensen's inequality, we get that

$$\varphi_k(\omega(t_k)) = \varphi_k\left(\omega\left(\sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i}\right)\right) \geq \sum_{i=1}^{\infty} \frac{\varphi_k(\omega(t_{k,i}))}{2^i}. \tag{3.3}$$

Employing (3.2) and (3.3) we obtain that

$$\begin{aligned}\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} &\geq \sum_{k=1}^{\infty} \lambda_k^{-1} \sum_{i=1}^{\infty} \varphi_k(\omega(t_{k,i}))2^{-i} \\ &= \sum_{i=1}^{\infty} 2^{-i} \sum_{k=1}^{\infty} \varphi_k(\omega(t_{k,i}))\lambda_k^{-1} = \infty,\end{aligned}$$

and this contradicts (2.1).

This contradiction proves (3.1). □

4. Proof of Theorem 2.1

Necessity. Suppose that $H^\omega \subset \Lambda\{\varphi_k\}BV$, but there exists a sequence $t = \{t_k\} \in T$ such that

$$\sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} = \infty. \tag{4.1}$$

Then, applying Lemma 3.2 with this sequence $t = \{t_k\} \in T$ and $\omega(\delta)$, we obtain that there exists a function $f \in H^\omega$ such that

$$|f(x_{2k-1}) - f(x_{2k-2})| = \omega(t_k) \text{ for all } k \in \mathbb{N}.$$

Hence, by (4.1), we get that

$$\sum_{k=1}^{N} \varphi_k(|f(x_{2k-1}) - f(x_{2k-2})|)\lambda_k^{-1} = \sum_{k=1}^{N} \varphi_k(\omega(t_k))\lambda_k^{-1} \to \infty,$$

that is, (1.1) does not hold if $b_k = x_{2k-1}$ and $a_k = x_{2k-2}$, thus f does not belong to the set $\Lambda\{\varphi_k\}BV$.

This and the assumption $H^\omega \subset \Lambda\{\varphi_k\}BV$ contradict, whence the necessity of (2.1) follows.

Sufficiency. The condition (2.1), by Lemma 3.3, implies (3.1). If we consider a system of nonoverlapping subintervals (a_k, b_k) of $[0,1]$ and take $t_k := (b_k - a_k)$, then $t := \{t_k\} \in T$, consequently for this t (3.1) holds. Thus, if $f \in H^\omega$, we always have that

$$\sum_{k=1}^{N} \varphi_k(|f(b_k) - f(a_k)|)\lambda_k^{-1} \leq K \sum_{k=1}^{N} \varphi_k(\omega(b_k - a_k))\lambda_k^{-1} \leq KM,$$

and this shows that $f \in \Lambda\{\varphi_k\}BV$.

The proof is complete.

REFERENCES

[1] A.V. EFIMOV, Linear methods of approximation of continuous periodic functions, *Mat. Sb.*, **54** (1961), 51–90 (in Russian).

[2] U. GOGINAVA, On the approximation properties of partial sums of trigonometric Fourier series, *East J. on Approximations*, **8** (2002), 403–420.

[3] H. KITA AND K. YONEDA, A generalization of bounded variation, *Acta Math. Hungar.*, **56** (1990), 229–238.

[4] L. LEINDLER, A note on embedding of classes H^ω, *Analysis Math.*, **27** (2001), 71–76.

[5] M.V. MEDVEDEVA, On embedding classes H^ω, *Mat. Zametki*, **64**(5) (1998), 713–719 (in Russian).

[6] L.C. YOUNG, Sur une généralization de la notion de variation de Wiener et sur la convergence des séries de Fourier, *C.R. Acad. Sci. Paris*, **204** (1937), 470–472.