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ON EMBEDDING OF THE CLASS  $H^\omega$

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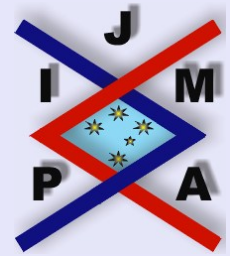


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## Abstract

In [4] we extended an interesting theorem of Medvedeva [5] pertaining to the embedding relation  $H^\omega \subset \Lambda BV$ , where  $\Lambda BV$  denotes the set of functions of  $\Lambda$ -bounded variation, which is encountered in the theory of Fourier trigonometric series. Now we give a further generalization of our result. Our new theorem tries to unify the notion of  $\varphi$ -variation due to Young [6], and that of the generalized Wiener class  $BV(p(n) \uparrow)$  due to Kita and Yoneda [3]. For further references we refer to the paper by Goginava [2].

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# 1. Introduction

Let  $\omega(\delta)$  be a nondecreasing continuous function on the interval  $[0, 1]$  having the following properties:

$$\omega(0) = 0, \omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2) \text{ for } 0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 1.$$

Such a function is called a modulus of continuity, and it will be denoted by  $\omega(\delta) \in \Omega$ .

The modulus of continuity of a continuous function  $f$  will be denoted by  $\omega(f; \delta)$ , that is,

$$\omega(f; \delta) := \sup_{\substack{0 \leq h \leq \delta \\ 0 \leq x \leq 1-h}} |f(x+h) - f(x)|.$$

As usual, set

$$H^\omega := \{f \in C : \omega(f; \delta) = O(\omega(\delta))\}.$$

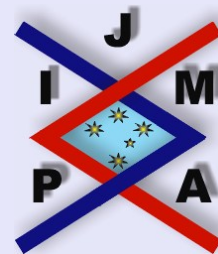
If  $\omega(\delta) = \delta^\alpha$ ,  $0 < \alpha \leq 1$  we write  $H^\alpha$  instead of  $H^{\delta^\alpha}$ .

Finally we define a new class of real functions  $f : [0, 1] \rightarrow \mathbb{R}$ . For every  $k \in \mathbb{N}$  let  $\varphi_k : [0, \infty) \rightarrow \mathbb{R}$  be a nondecreasing function with  $\varphi_k(0) = 0$ ; and let  $\Lambda := \{\lambda_k\}$  be a nondecreasing sequence of positive numbers such that

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty.$$

If a function  $f : [0, 1] \rightarrow \mathbb{R}$  satisfies the condition

$$(1.1) \quad \sup \sum_{k=1}^N \varphi_k(|f(b_k) - f(a_k)|) \lambda_k^{-1} < \infty,$$



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where the supremum is extended over all systems of nonoverlapping subintervals  $(a_k, b_k)$  of  $[0, 1]$ , then  $f$  is said to be of  $\Lambda\{\varphi_k\}$ -bounded variation, and this fact is denoted by  $f \in \Lambda\{\varphi_k\}BV$ . In the special cases when all  $\varphi_k(x) = \varphi(x)$ , we write  $f \in \Lambda_\varphi BV$  (see [4]), and if  $\varphi(x) = x^p$  we use the notation  $f \in \Lambda_p BV$ , and when  $p = 1$ , simply  $f \in \Lambda BV$  (see [5]). In the case  $\lambda_k = 1$  and  $\varphi_k(x) = \varphi(x)$  for all  $k$ , then we get the class  $V_\varphi$  due to Young [6], finally if  $\lambda_k = 1$  and  $\varphi_k(x) = x^{p_k}$ ,  $p_k \uparrow$ , we get a class similar to  $BV(p(n) \uparrow)$  (see [3]).

Medvedeva [5] proved the following useful theorem, among others.

**Theorem 1.1.** *The embedding relation  $H^\omega \subset \Lambda BV$  holds if and only if*

$$\sum_{k=1}^{\infty} \omega(t_k) \lambda_k^{-1} < \infty$$

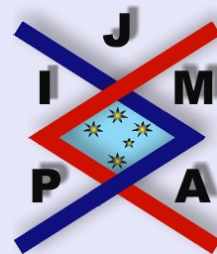
for any sequence  $\{t_k\}$  satisfying the conditions:

$$(1.2) \quad t_k \geq 0, \quad \sum_{k=1}^{\infty} t_k \leq 1.$$

In the sequel, the fact that a sequence  $t := \{t_k\}$  has the properties (1.2) will be denoted by  $t \in T$ .  $K$  and  $K_i$  will denote positive constants, not necessarily the same at each occurrence.

Among others, in [4] we showed that if  $0 < \alpha \leq 1$  and  $p\alpha \geq 1$  then  $H^\alpha \subset \Lambda_p BV$  always holds, furthermore that if  $0 < p < 1/\alpha$ , then  $H^\alpha \subset \Lambda_p BV$  is fulfilled if and only if for any  $t \in T$ ,

$$\sum_{k=1}^{\infty} t_k^{\alpha p} \lambda_k^{-1} < \infty.$$



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If  $\omega(\delta)$  is a general modulus of continuity then for  $0 < p < 1$  we verified that  $H^\omega \subset \Lambda_p BV$  holds if and only if for any  $t \in T$

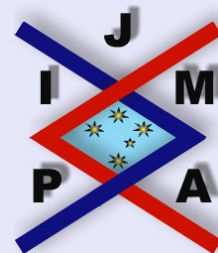
$$(1.3) \quad \sum_{k=1}^{\infty} \omega(t_k)^p \lambda_k^{-1} < \infty.$$

These latter two results are immediate consequences of the following theorem of [4].

**Theorem 1.2.** *Assume that  $\varphi(x)$  is a function such that  $\varphi(\omega(\delta)) \in \Omega$ . Then  $H^\omega \subset \Lambda_\varphi BV$  holds if and only if for any  $t \in T$*

$$\sum_{k=1}^{\infty} \varphi(\omega(t_k)) \lambda_k^{-1} < \infty.$$

**Remark 1.1.** *It would be of interest to mention that by Theorem 1.2 the restriction  $0 < p < 1$  claimed above, can be replaced by the weaker condition  $\omega(\delta)^p \in \Omega$ , and then the embedding relation  $H^\omega \subset \Lambda_p BV$  also holds if and only if (1.3) is true.*




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## 2. Results

Our new theorem tries to unify and generalize all of the former results.

**Theorem 2.1.** *Assume that  $\omega(t) \in \Omega$  and for every  $k \in \mathbb{N}$ ,  $\varphi_k(\omega(\delta)) \in \Omega$ . Then the embedding relation  $H^\omega \subset \Lambda\{\varphi_k\}BV$  holds if and only if for any  $t \in T$*

$$(2.1) \quad \sum_{k=1}^{\infty} \varphi_k(\omega(t_k)) \lambda_k^{-1} < \infty.$$

Our theorem plainly yields the following assertion.

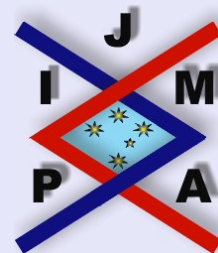
**Corollary 2.2.** *If for all  $k \in \mathbb{N}$ ,  $p_k > 0$  and  $\omega(\delta)^{p_k} \in \Omega$ , that is, if  $\varphi_k(x) = x^{p_k}$ , then  $H^\omega \subset \Lambda\{x^{p_k}\}BV$  holds if and only if for any  $t \in T$*

$$(2.2) \quad \sum_{k=1}^{\infty} \omega(t_k)^{p_k} \lambda_k^{-1} < \infty.$$

*It is also obvious that if  $\omega(\delta) = \delta^\alpha$ ,  $0 < \alpha \leq 1$ , then (2.1) and (2.2) reduce to*

$$\sum_{k=1}^{\infty} \varphi_k(t_k^\alpha) \lambda_k^{-1} < \infty \text{ and } \sum_{k=1}^{\infty} t_k^{\alpha p_k} \lambda_k^{-1} < \infty,$$

*respectively.*



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### 3. Lemmas

In the proof we shall use the following three lemmas.

**Lemma 3.1** ([1, p. 78]). *If  $\omega(\delta) \in \Omega$  then there exists a concave function  $\omega^*(\delta)$  such that*

$$\omega(\delta) \leq \omega^*(\delta) \leq 2\omega(\delta).$$

**Lemma 3.2.** *If  $\omega(\delta) \in \Omega$  and  $t = \{t_k\} \in T$ , then there exists a function  $f \in H^\omega$  such that if*

$$x_0 = 0, \quad x_1 = \frac{t_1}{2},$$

$$x_{2n} = \sum_{i=1}^n t_i \text{ and } x_{2n+1} = x_{2n} + \frac{t_{n+1}}{2}, \quad n \geq 1,$$

then

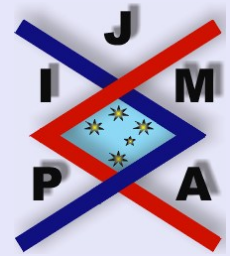
$$f(x_{2n}) = 0 \text{ and } f(x_{2n+1}) = \omega(t_{n+1}) \text{ for all } n \geq 0.$$

A concrete function with these properties is given in [5].

**Lemma 3.3.** *If  $\omega(t) \in \Omega$  and for all  $k \in \mathbb{N}$ ,  $\varphi_k(\omega(t)) \in \Omega$  also holds, furthermore for any  $t \in T$  the condition (2.1) stays, then there exists a positive number  $M$  such that for any  $t \in T$*

$$(3.1) \quad \sum_{k=1}^{\infty} \varphi_k(\omega(t_k)) \lambda_k^{-1} \leq M$$

holds.



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*Proof of Lemma 3.3.* The proof follows the lines given in the proof of Theorem 2 emerging in [5]. Without loss of generality, due to Lemma 3.1, we can assume that, for every  $k$ , the functions  $\varphi_k(\omega(\delta))$  are concave moduli of continuity.

Indirectly, let us suppose that there is no number  $M$  with property (3.1). Then for any  $i \in \mathbb{N}$  there exists a sequence  $t^{(i)} := \{t_{k,i}\} \in T$  such that

$$(3.2) \quad 2^i < \sum_{k=1}^{\infty} \varphi_k(\omega(t_{k,i})) \lambda_k^{-1} < \infty.$$

Now define

$$t_k := \sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i}.$$

It is easy to see that  $t := \{t_k\} \in T$ , and thus (2.1) also holds.

Since every  $\varphi_k(\omega(\omega))$  is concave, thus by Jensen's inequality, we get that

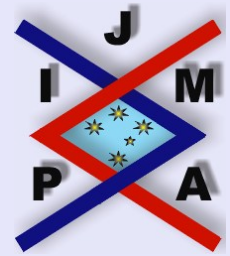
$$(3.3) \quad \varphi_k(\omega(t_k)) = \varphi_k \left( \omega \left( \sum_{i=1}^{\infty} \frac{t_{k,i}}{2^i} \right) \right) \geq \sum_{i=1}^{\infty} \frac{\varphi_k(\omega(t_{k,i}))}{2^i}.$$

Employing (3.2) and (3.3) we obtain that

$$\begin{aligned} \sum_{k=1}^{\infty} \varphi_k(\omega(t_k)) \lambda_k^{-1} &\geq \sum_{k=1}^{\infty} \lambda_k^{-1} \sum_{i=1}^{\infty} \varphi_k(\omega(t_{k,i})) 2^{-i} \\ &= \sum_{i=1}^{\infty} 2^{-i} \sum_{k=1}^{\infty} \varphi_k(\omega(t_{k,i})) \lambda_k^{-1} = \infty, \end{aligned}$$

and this contradicts (2.1).

This contradiction proves (3.1). □



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## 4. Proof of Theorem 2.1

*Necessity.* Suppose that  $H^\omega \subset \Lambda\{\varphi_k\}BV$ , but there exists a sequence  $t = \{t_k\} \in T$  such that

$$(4.1) \quad \sum_{k=1}^{\infty} \varphi_k(\omega(t_k))\lambda_k^{-1} = \infty.$$

Then, applying Lemma 3.2 with this sequence  $t = \{t_k\} \in T$  and  $\omega(\delta)$ , we obtain that there exists a function  $f \in H^\omega$  such that

$$|f(x_{2k-1}) - f(x_{2k-2})| = \omega(t_k) \text{ for all } k \in \mathbb{N}.$$

Hence, by (4.1), we get that

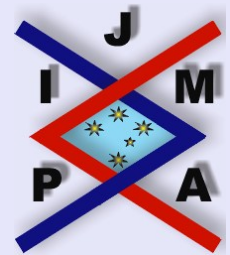
$$\sum_{k=1}^N \varphi_k(|f(x_{2k-1}) - f(x_{2k-2})|)\lambda_k^{-1} = \sum_{k=1}^N \varphi_k(\omega(t_k))\lambda_k^{-1} \rightarrow \infty,$$

that is, (1.1) does not hold if  $b_k = x_{2k-1}$  and  $a_k = x_{2k-2}$ , thus  $f$  does not belong to the set  $\Lambda\{\varphi_k\}BV$ .

This and the assumption  $H^\omega \subset \Lambda\{\varphi_k\}BV$  contradict, whence the necessity of (2.1) follows.

*Sufficiency.* The condition (2.1), by Lemma 3.3, implies (3.1). If we consider a system of nonoverlapping subintervals  $(a_k, b_k)$  of  $[0, 1]$  and take  $t_k := (b_k - a_k)$ , then  $t := \{t_k\} \in T$ , consequently for this  $t$  (3.1) holds. Thus, if  $f \in H^\omega$ , we always have that

$$\sum_{k=1}^N \varphi_k(|f(b_k) - f(a_k)|)\lambda_k^{-1} \leq K \sum_{k=1}^N \varphi_k(\omega(b_k - a_k))\lambda_k^{-1} \leq KM,$$



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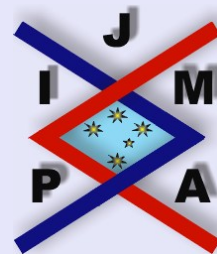
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and this shows that  $f \in \Lambda\{\varphi_k\}BV$ .

The proof is complete.



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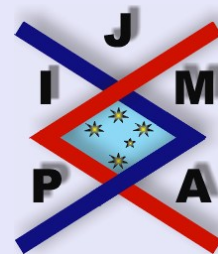
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