ON A RESULT OF TOHGE CONCERNING THE UNICITY OF MEROMORPHIC FUNCTIONS

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Abstract: In this paper we prove some uniqueness theorems of meromorphic functions which improve a result of Tohge and answer a question given by him. Furthermore, an example shows that the conditions of our results are sharp.

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1. Introduction, Definitions and Results

Let $f(z)$ be a nonconstant meromorphic function in the complex plane C. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f)$, $N(r,f)$, and $m(r,f)$ (see, e.g., [1]). In this paper, we use $N_{k)}(r,1/(f-a))$ to denote the counting function of a-points of f with multiplicities less than or equal to k, and $N_{(k}(r,1/(f-a))$ the counting function of a-points of f with multiplicities greater than or equal to k. We also use $\overline{N}_{k)}(r,1/(f-a))$ and $\overline{N}_{(k}(r,1/(f-a))$ to denote the corresponding reduced counting functions, respectively (see [2]). The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f)=o(T(r,f))$ as $r\to\infty$ possibly outside a set of r of finite linear measure.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and a be a complex number. If the zeros of $f-a$ and $g-a$ have the same zeros counting multiplicities (ignoring multiplicities), then we say that f and g share the value a CM (IM).

Let $S_0(f=a=g)$ be the set of all common zeros of $f(z)-a$ and $g(z)-a$ ignoring multiplicities, $S_E(f=a=g)$ be the set of all common zeros of $f(z)-a$ and $g(z)-a$ with the same multiplicities. Denote by $\overline{N}_0(r,f=a=g)$, $\overline{N}_E(r,f=a=g)$ the reduced counting functions of f and g corresponding to the sets $S_0(f=a=g)$ and $S_E(f=a=g)$, respectively. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_0(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a IM*. If

$$\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}\left(r,\frac{1}{g-a}\right)-2\overline{N}_E(r,f=a=g)=S(r,f)+S(r,g),$$

then we say that f and g share a CM*.

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Let k be a positive integer or infinity. We denote by $\overline{E}_{k)}(a,f)$ the set of a-points of f with multiplicities less than or equal to k (ignoring multiplicities).

In 1988, Tohge [3] proved the following result.

Theorem A ([3]). *Let f and g be two nonconstant meromorphic functions sharing 0, 1, ∞ CM, and f', g' share 0 CM. Then f and g satisfy one of the following relations:*

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f-1)(g-1) \equiv 1$,

(iv) $f+g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f-1 \equiv c(g-1)$,

(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,

where c $(\neq 0,1)$ is a constant.

In the same paper, Tohge [3] suggested the following problem: *Is it possible to weaken the restriction of CM sharing in Theorem A?*

In 2000, Al-Khaladi [4] – [5] dealt with this problem and proved the following theorems, which are improvements of Theorem A.

Theorem B ([4]). *Let f and g be two nonconstant meromorphic functions sharing 0, 1, ∞ CM, and f', g' share 0 IM. Then the conclusions of Theorem A still hold.*

Theorem C ([5]). *Let f and g be two nonconstant meromorphic functions sharing 0, ∞ CM, and f', g' share 0 IM. If $\overline{E}_{k)}(1,f) = \overline{E}_{k)}(1,g)$, where k is a positive integer or infinity, then the conclusions of Theorem A still hold.*

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Now we explain the notion of weighted sharing as introduced in [6] – [7].

Definition 1.1 ([6] – [7]). *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \bigcup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity m $(\le k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\le k)$ and z_0 is a zero of $f-a$ with multiplicity m $(> k)$ if and only if it is a zero of $g-a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In particular, if f, g share a value a IM* or CM*, then we say that f, g share $(a, 0)^*$ or $(a, \infty)^*$ respectively (see [8]).

Definition 1.2 ([8]). *For $a \in \mathbb{C} \bigcup \{\infty\}$, we put*

$$\delta_{(p}(a, f) = 1 - \limsup_{r \to \infty} \frac{N_{(p}\left(r, \frac{1}{f-a}\right)}{T(r, f)},$$

where p is a positive number.

In 2005, the present author etc. [8] and Lahiri [9] also improved Theorem A and obtained the following results, respectively.

Theorem D ([8]). *Let f and g be two nonconstant meromorphic functions sharing $(0, 1)$, $(1, \infty)$, (∞, ∞), and f', g' share $(0, 0)^*$. If $\delta_{(2}(0, f) > 1/2$, then the conclusions of Theorem A still hold.*

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Theorem E ([9]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, and (∞,k), where k, m are positive integers or infinities satisfying $(m-1)(km-1)>(1+m)^2$. If $\overline{E}_{1)}(0,f')\subseteq\overline{E}_{\infty)}(0,g')$ and $\overline{E}_{1)}(0,g')\subseteq\overline{E}_{\infty)}(0,f')$, then the conclusions of Theorem A still hold.*

In this paper, we shall prove the following theorems, which improve and supplement the above theorems.

Theorem 1.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and k_j $(j=1,2,3)$ are positive integers satisfying*

$$k_1k_2k_3>k_1+k_2+k_3+2. \tag{1.1}$$

If $\overline{E}_{1)}(0,f')\subseteq\overline{E}_{\infty)}(0,g')$ and $\overline{E}_{1)}(0,g')\subseteq\overline{E}_{\infty)}(0,f')$, then f and g satisfy one of the following relations:

(i) $f\equiv g$,

(ii) $fg\equiv 1$,

(iii) $(f-1)(g-1)\equiv 1$,

(iv) $f+g\equiv 1$,

(v) $f\equiv cg$,

(vi) $f-1\equiv c(g-1)$,

(vii) $[(c-1)f+1][(c-1)g-c]\equiv -c$,

where c $(\neq 0,1)$ is a constant.

From Theorem 1.3, we immediately deduce the following corollary.

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Corollary 1.4. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying one of the following relations:*

(i) $k_1 \geq 1$, $k_2 \geq 3$, *and* $k_3 \geq 4$,

(ii) $k_1 \geq 2$, $k_2 \geq 2$, *and* $k_3 \geq 3$,

(iii) $k_1 \geq 1$, $k_2 \geq 2$, *and* $k_3 \geq 6$.

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f - 1)(g - 1) \equiv 1$,

(iv) $f + g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g - 1)$,

(vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

Theorem 1.5. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying (1.1). If*

$$N_{1)}\left(r, \frac{1}{f'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) < (\lambda + o(1))T(r), \quad (r \in I), \tag{1.2}$$

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where $0 < \lambda < 1/3$, $T(r) = \max\{T(r,f), T(r,g)\}$, *and* I *is a set of infinite linear measure, then* f *and* g *satisfy one of the following relations: (i)* $f \equiv g$, *(ii)* $fg \equiv 1$, *(iii)* $(f-1)(g-1) \equiv 1$, *(iv)* $f + g \equiv 1$, *(v)* $f \equiv cg$, *(vi)* $f - 1 \equiv c(g-1)$, *(vii)* $[(c-1)f+1][(c-1)g-c] \equiv -c$, *where* $c\ (\neq 0, 1)$ *is a constant.*

By Theorem 1.5, we instantly derive the following corollary.

Corollary 1.6. *Let* f *and* g *be two nonconstant meromorphic functions sharing* (a_1, k_1), (a_2, k_2), *and* (a_3, k_3), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, *and* k_j $(j = 1, 2, 3)$ *are positive integers satisfying one of the following relations:*

(i) $k_1 \geq 1$, $k_2 \geq 3$, *and* $k_3 \geq 4$,

(ii) $k_1 \geq 2$, $k_2 \geq 2$, *and* $k_3 \geq 3$,

(iii) $k_1 \geq 1$, $k_2 \geq 2$, *and* $k_3 \geq 6$.

If (1.2) *holds, then* f *and* g *satisfy one of the following relations:*

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f-1)(g-1) \equiv 1$,

(iv) $f + g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g-1)$,

(vii) $[(c-1)f+1][(c-1)g-c] \equiv -c$,

where $c\ (\neq 0, 1)$ *is a constant.*

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The following example shows that any one of k_j $(j = 1, 2, 3)$ in Theorem 1.3, Corollary 1.4, Theorem 1.5 and Corollary 1.6 cannot be equal to 0.

Example 1.1. Let $f = (e^z - 1)^{-2}$ and $g = (e^z - 1)^{-1}$. Then f and g share $(0, \infty)$, $(1, \infty)$, $(\infty, 0)$, and f', g' share $(0, \infty)$. However, f and g do not satisfy any one of the relations given in Theorem 1.3, Corollary 1.4, Theorem 1.5 and Corollary 1.6.

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2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1 ([10]). *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(1,0)$, and $(\infty,0)$. Then*

$$T(r,f) \leq 3T(r,g) + S(r,f), \qquad T(r,g) \leq 3T(r,f) + S(r,g),$$

$$S(r,f) = S(r,g) := S(r).$$

Proof. Note that f and g share $(0,0)$, $(1,0)$, and $(\infty,0)$. By the second fundamental theorem, we can easily obtain the conclusion of Lemma 2.1. □

The second lemma is due to Yi [11], which plays an important role in the proof.

Lemma 2.2 ([11]). *Let f and g be two distinct nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\} = \{0,1,\infty\}$, and k_j $(j = 1,2,3)$ are positive integers satisfying (1.1). Then*

$$\overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) = S(r),$$

the same identity holds for g.

Lemma 2.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\} = \{0,1,\infty\}$, and k_j $(j = 1,2,3)$ are positive integers satisfying (1.1). If*

$$\alpha = \frac{g}{f}, \tag{2.1}$$

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$$(2.2)\qquad \beta = \frac{f-1}{g-1},$$

then

$$\overline{N}\left(r,\frac{1}{\alpha}\right) = \overline{N}(r,\alpha) = \overline{N}\left(r,\frac{1}{\beta}\right) = \overline{N}(r,\beta) = S(r).$$

Proof. If α or β is a constant, then the result is obvious. Next we suppose that α and β are nonconstant. Since f and g share (a_1,k_1), (a_2,k_2), and (a_3,k_3), by (2.1), (2.2), and Lemma 2.2 we have

$$\overline{N}\left(r,\frac{1}{\alpha}\right) \le \overline{N}_{(2}\left(r,\frac{1}{g}\right) + \overline{N}_{(2}(r,f) = S(r),$$

$$\overline{N}(r,\alpha) \le \overline{N}_{(2}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}(r,g) = S(r),$$

$$\overline{N}(r,\frac{1}{\beta}) \le \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}(r,g) = S(r),$$

$$\overline{N}(r,\beta) \le \overline{N}_{(2}\left(r,\frac{1}{g-1}\right) + \overline{N}_{(2}(r,f) = S(r),$$

which completes the proof of the lemma. □

Lemma 2.4. *Let f and g be two distinct nonconstant meromorphic functions sharing (a_1,k_1), (a_2,k_2), and (a_3,k_3), where $\{a_1,a_2,a_3\} = \{0,1,\infty\}$, and k_j $(j=1,2,3)$ are positive integers satisfying (1.1). If f is not a fractional linear transformation of g, then*

$$\overline{N}_{(2}\left(r,\frac{1}{f'}\right) = S(r), \qquad \overline{N}_{(2}\left(r,\frac{1}{g'}\right) = S(r).$$

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Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Let α and β be given by (2.1) and (2.2). From (2.1) and (2.2), we have

$$f = \frac{1-\beta}{1-\alpha\beta}, \tag{2.3}$$

$$g = \frac{(1-\beta)\alpha}{1-\alpha\beta}. \tag{2.4}$$

Since f is not a fractional linear transformation of g, we know that α, β, and $\alpha\beta$ are nonconstant. Let

$$h := \frac{\alpha\beta'}{\alpha\beta' + \alpha'\beta} = \frac{\beta'/\beta}{\alpha'/\alpha + \beta'/\beta}. \tag{2.5}$$

Then we have $h \not\equiv 0, 1$. Note that

$$N\left(r, \frac{\alpha'}{\alpha}\right) = \overline{N}\left(r, \frac{1}{\alpha}\right) + \overline{N}(r, \alpha),$$

$$N\left(r, \frac{\beta'}{\beta}\right) = \overline{N}\left(r, \frac{1}{\beta}\right) + \overline{N}(r, \beta).$$

From this and Lemma 2.3, we get

$$T\left(r, \frac{\alpha'}{\alpha}\right) = T\left(r, \frac{\beta'}{\beta}\right) = S(r), \tag{2.6}$$

and so

$$T(r, h) = S(r). \tag{2.7}$$

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By (2.3), we get

$$f - h = \frac{(1-\beta) - h(1-\alpha\beta)}{1-\alpha\beta}. \tag{2.8}$$

Let

$$F := (f-h)(1-\alpha\beta) = (1-\beta) - h(1-\alpha\beta). \tag{2.9}$$

From (2.5) and (2.9), we have

$$\begin{aligned} \frac{F'}{F} - \frac{\beta'}{\beta} &= \frac{-\beta' - h'(1-\alpha\beta) + \alpha\beta' - \beta' F/\beta}{F} \\ &= \frac{1}{f-h}\left[\frac{\beta'}{\beta}(h-1) - h'\right]. \end{aligned} \tag{2.10}$$

If $\beta'(h-1)/\beta - h' \equiv 0$, then from this and (2.10), we get

$$h = c_1\beta + 1, \tag{2.11}$$

and so $F'/F - \beta'/\beta \equiv 0$, i.e.,

$$F = c_2\beta, \tag{2.12}$$

where c_1, c_2 are nonzero constants. By (2.7), (2.11), and (2.12), we have

$$T(r,F) = T(r,\beta) = S(r).$$

From this, (2.7), and (2.9), we get

$$T(r,\alpha) = S(r),$$

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and so $T(r,f) = S(r)$, which is impossible. Therefore $\beta'(h-1)/\beta - h' \not\equiv 0$. By (2.10), we have

$$\frac{1}{f-h} = \frac{F'/F - \beta'/\beta}{\beta'(h-1)/\beta - h'}. \tag{2.13}$$

From (2.6), (2.7), and (2.13), we get

$$m\left(r, \frac{1}{f-h}\right) \le m\left(r, \frac{F'}{F}\right) + S(r) = S(r). \tag{2.14}$$

Since F'/F and β'/β have only simple poles, it follows again from (2.6), (2.7), and (2.13) that

$$\begin{aligned} N_{(2}\left(r, \frac{1}{f-h}\right) &\le 2N\left(r, \frac{1}{\beta'(h-1)/\beta - h'}\right) + S(r) \\ &\le 2T\left(r, \frac{\beta'(h-1)}{\beta} - h'\right) + S(r) \\ &\le 2T\left(r, \frac{\beta'}{\beta}\right) + 2T(r,h) + 2T(r,h') + S(r) \\ &\le S(r), \end{aligned}$$

i.e.,

$$N_{(2}\left(r, \frac{1}{f-h}\right) = S(r). \tag{2.15}$$

By (2.2) and (2.4), we have

$$\frac{g-f}{g-1} = 1 - \beta,$$

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$$\frac{g'}{g} = \frac{\alpha'(1-\alpha\beta) + (\alpha-1)(\alpha\beta' + \alpha'\beta)}{\alpha(1-\beta)(1-\alpha\beta)}.$$

Therefore

$$\frac{g'(g-f)}{g(g-1)} = \frac{(1-\beta)(\alpha\beta' + \alpha'\beta) - \alpha\beta'(1-\alpha\beta)}{\alpha\beta(1-\alpha\beta)}. \tag{2.16}$$

From (2.5) and (2.8), we get

$$(f-h)\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right) = \frac{(1-\beta)(\alpha\beta' + \alpha'\beta) - \alpha\beta'(1-\alpha\beta)}{\alpha\beta(1-\alpha\beta)}. \tag{2.17}$$

By (2.16) and (2.17), we have

$$\frac{g'(g-f)}{g(g-1)} = (f-h)\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right). \tag{2.18}$$

Let $N_0^{(2}(r, 1/g')$ denote the counting function corresponding to multiple zeros of g' that are not zeros of g and $g-1$. Then from (2.15) and (2.18), we get

$$N_0^{(2}\left(r, \frac{1}{g'}\right) \le N_{(2}\left(r, \frac{1}{f-h}\right) + S(r) \le S(r).$$

From this and Lemma 2.2, we have

$$\overline{N}_{(2}\left(r, \frac{1}{g'}\right) \le N_0^{(2}\left(r, \frac{1}{g'}\right) + \overline{N}_{(2}\left(r, \frac{1}{g}\right) + \overline{N}_{(2}\left(r, \frac{1}{g-1}\right) \le S(r),$$

i.e.,

$$\overline{N}_{(2}\left(r, \frac{1}{g'}\right) = S(r).$$

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Similarly, we can prove

$$\overline{N}_{(2}\left(r, \frac{1}{f'}\right) = S(r),$$

which also completes the proof of Lemma 2.4. □

Lemma 2.5. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, k_3), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_j $(j = 1, 2, 3)$ are positive integers satisfying (1.1). If f is a fractional linear transformation of g, then f and g satisfy one of the following relations:*

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f - 1)(g - 1) \equiv 1$,

(iv) $f + g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g - 1)$,

(vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

Proof. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Since f is a fractional linear transformation of g, we can suppose that

$$f = \frac{Ag + B}{Cg + D},$$

where A, B, C, D are constants such that $AD - BC \neq 0$.

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If $f \equiv g$, then the relation (i) holds. Next we assume that $f \not\equiv g$ and discuss the following cases.

Case 1 If none of 0, 1, and ∞ are Picard's exceptional values of f and g, then $f \equiv g$, which contradicts the assumption.

Case 2 If 0 and 1 are all Picard's exceptional values of f and g, then $f = \alpha g + \beta = \alpha(g + \beta/\alpha)$, where α $(\neq 0)$, β are constants. Since $f \neq 0$, it follows that $\beta/\alpha = 0$ or -1.

Subcase 2.1 If $\beta = 0$, then $f = \alpha g$, i.e., $f - 1 = \alpha(g - 1/\alpha)$. Since $f \neq 1$, it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.

Subcase 2.2 If $\beta/\alpha = -1$, then $f = \alpha g - \alpha$, i.e., $f - 1 = \alpha(g - (\alpha+1)/\alpha)$. Since $f \neq 1$, it follows that $\alpha = -1$. Thus $f \equiv -g + 1$, which implies the relation (iv).

Case 3 If 1 and ∞ are all Picard's exceptional values of f and g, then $f = Ag/(Cg + D)$, where A $(\neq 0)$, D $(\neq 0)$ are constants.

Subcase 3.1 If $C = 0$, then $f = \alpha g$, i.e., $f - 1 = \alpha(g - 1/\alpha)$, where α $(\neq 0)$ is a constant. Since $f \neq 1$ and $g \neq 1, \infty$, it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.

Subcase 3.2 If $C \neq 0$, then $f = \alpha g/(g - 1)$, i.e., $f - 1 = ((\alpha - 1)g + 1)/(g - 1)$, where α $(\neq 0)$ is a constant. Since $f \neq 1$ and $g \neq 1, \infty$, it follows that $\alpha = 1$ and so $f - 1 \equiv 1/(g - 1)$. This is the relation (iii).

Case 4 If 0 and ∞ are all Picard's exceptional values of f and g, then $f = (Ag + B)/(Cg + D)$, where $A + B = C + D$.

Subcase 4.1 If $A = 0$, then $f = B/(Cg + D)$, where B $(\neq 0)$, C $(\neq 0)$ are constants. Since $f \neq \infty$ and $g \neq 0, \infty$, it follows that $D = 0$. Thus $fg \equiv 1$ because f and g share $(1, k_2)$. This is the relation (ii).

Subcase 4.2 If $A \neq 0$ and $C = 0$, then $f = \alpha g + \beta$, where α $(\neq 0)$, β are constants. Since $f \neq 0$ and $g \neq 0, \infty$, it follows that $\beta = 0$. Thus $f \equiv g$ because f and g share $(1, k_2)$. This is a contradiction.

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Subcase 4.3 If $A \neq 0$ and $C \neq 0$, then it follows that $B = D = 0$ because $f \neq 0, \infty$ and $g \neq 0, \infty$. Thus $f \equiv$ constant, which contradicts the assumption.

Case 5 If 0 is Picard's exceptional value of f and g but 1 and ∞ are not, then it follows that $C = 0$ because f and g share (∞, k_3). Thus $f = \alpha g + \beta$, where α $(\neq 0)$, β are constants such that $\alpha + \beta = 1$.

Subcase 5.1 If $\beta = 0$, then it follows that $\alpha = 1$ and so $f \equiv g$. This is a contradiction.

Subcase 5.2 If $\beta \neq 0$, then it follows that $\beta = 1 - \alpha$ and so $f \equiv \alpha g + 1 - \alpha$, where α $(\neq 0, 1)$ is a constant. This is the relation (vi).

Case 6 If 1 is Picard's exceptional value of f and g but 0 and ∞ are not, then it follows that $C = 0$ because f and g share (∞, k_3). Since f and g share $(0, k_1)$, it follows that $B = 0$ and so $f \equiv \alpha g$, where α $(\neq 0)$ is a constant. If $\alpha = 1$, then $f \equiv g$, which is a contradiction. Thus $f \equiv \alpha g$, where α $(\neq 0, 1)$ is a constant. This is the relation (v).

Case 7 If ∞ is Picard's exceptional value of f and g but 0 and 1 are not, then it follows that $B = 0$ and $A = C + D$ because f and g share $(0, k_1)$ and $(1, k_2)$. Thus $f = Ag/(Cg + D)$, where A $(\neq 0)$, D $(\neq 0)$ are constants.

Subcase 7.1 If $C = 0$, then it follows that $A = D$ because f and g share $(1, k_2)$. Thus $f \equiv g$, which is a contradiction.

Subcase 7.2 If $C \neq 0$, then it follows that $f = \alpha g/(g + \beta)$ and $\alpha = 1 + \beta$, where α $(\neq 0, 1)$, β are constants. Thus $f \equiv \alpha g/(g + \alpha - 1)$, i.e., $fg - (1 - \alpha)f - \alpha g \equiv 0$, which implies the relation (vii).

This completes the proof of Lemma 2.5. □

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3. Proofs of the Theorems

Proof of Theorem 1.3. Without loss of generality, we assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Otherwise, a fractional linear transformation will do. Let α and β be given by (2.1) and (2.2).

Suppose now that f is not a fractional linear transformation of g. Then from Lemma 2.4, we have

$$\overline{N}_{(2}\left(r, \frac{1}{f'}\right) = S(r), \qquad \overline{N}_{(2}\left(r, \frac{1}{g'}\right) = S(r). \tag{3.1}$$

By (2.1), we get

$$\frac{\alpha'}{\alpha} = \frac{g'}{g} - \frac{f'}{f},$$

i.e.,

$$\frac{\alpha'}{\alpha} f = \frac{f}{g} g' - f'. \tag{3.2}$$

Let z_0 be a simple zero of g' that is not a zero of f and g. Then it follows that z_0 is a simple zero of f' because $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$. Again from (3.2), we deduce that z_0 is a zero of α'/α. On the other hand, the process of proving Lemma 2.4 shows that

$$T\left(r, \frac{\alpha'}{\alpha}\right) = T\left(r, \frac{\beta'}{\beta}\right) = S(r).$$

From this, (3.1), and Lemma 2.2, we have

$$\overline{N}\left(r, \frac{1}{g'}\right) = \overline{N}_{(2}\left(r, \frac{1}{g'}\right) + N_{1)}\left(r, \frac{1}{g'}\right) \tag{3.3}$$

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$$\leq N\left(r, \frac{\alpha'}{\alpha}\right) + \overline{N}_{(2}\left(r, \frac{1}{g}\right) + S(r)$$
$$\leq S(r).$$

Similarly, we can prove

$$\overline{N}\left(r, \frac{1}{f'}\right) = S(r). \tag{3.4}$$

Let

$$\Delta_1 := \left(\frac{f''}{f'} - \frac{2f'}{f}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g}\right).$$

If $\Delta_1 \equiv 0$, then by integration we obtain

$$\frac{1}{f} = \frac{c}{g} + d,$$

i.e.,

$$f = \frac{g}{c + dg},$$

where $c\ (\neq 0)$, d are constants. Thus f is a fractional linear transformation of g, which contradicts the assumption. Hence $\Delta_1 \not\equiv 0$.

Since f and g share $(0, k_1)$, it follows that a simple zero of f is a simple zero of g and conversely. Let z_0 be a simple zero of f and g. Then in some neighborhood of z_0, we get $\Delta_1 = (z - z_0)\gamma(z)$, where γ is analytic at z_0. Thus by (3.3), (3.4), and Lemma 2.2, we get

$$N_{1)}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{\Delta_1}\right)$$
$$\leq N(r, \Delta_1) + S(r)$$

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$$\le \overline{N}\left(r,\frac{1}{f'}\right)+\overline{N}\left(r,\frac{1}{g'}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)$$
$$+\overline{N}_{(2}\left(r,\frac{1}{g}\right)+\overline{N}_{(2}(r,f)+\overline{N}_{(2}(r,g)+S(r)$$
$$\le S(r),$$

and so

$$\overline{N}\left(r,\frac{1}{f}\right)=N_{1)}\left(r,\frac{1}{f}\right)+\overline{N}_{(2}\left(r,\frac{1}{f}\right)=S(r). \tag{3.5}$$

Let

$$\Delta_2 := \left(\frac{f''}{f'}-\frac{2f'}{f-1}\right)-\left(\frac{g''}{g'}-\frac{2g'}{g-1}\right),$$

and

$$\Delta_3 := \frac{f''}{f'}-\frac{g''}{g'}.$$

In the same manner as the above, we can obtain

$$\overline{N}\left(r,\frac{1}{f-1}\right)=S(r), \tag{3.6}$$

and

$$\overline{N}(r,f)=S(r). \tag{3.7}$$

From (3.5), (3.6), (3.7), and the second fundamental theorem, we have

$$T(r,f)\le \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r)\le S(r),$$

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which is a contradiction. Therefore f is a fractional linear transformation of g. Again from Lemma 2.5, we obtain the conclusion of Theorem 1.3. □

Proof of Theorem 1.5. Likewise, we can assume that $a_1 = 0$, $a_2 = 1$, and $a_3 = \infty$. Suppose now that f is not a fractional linear transformation of g.

Let

$$T(r) = \begin{cases} T(r,f), & \text{for} \quad r \in I_1, \\ T(r,g), & \text{for} \quad r \in I_2, \end{cases} \tag{3.8}$$

where

$$I = I_1 \cup I_2. \tag{3.9}$$

Note that I is a set of infinite linear measure of $(0,\infty)$. We can see by (3.9) that I_1 is a set of infinite linear measure of $(0,\infty)$ or I_2 is a set of infinite linear measure of $(0,\infty)$. Without loss of generality, we assume that I_1 is a set of infinite linear measure of $(0,\infty)$. Then by (3.8), we have

$$T(r) = T(r,f). \tag{3.10}$$

Let Δ_1, Δ_2, and Δ_3 be defined as in Theorem 1.3. Similar to the proof of (3.5), (3.6), and (3.7) in Theorem 1.3, we easily get

$$\begin{aligned} \overline{N}\left(r,\frac{1}{f}\right) &= N_{1)}\left(r,\frac{1}{f}\right) + \overline{N}_{(2}\left(r,\frac{1}{f}\right) \\ &\le N_{1)}\left(r,\frac{1}{f'}\right) + N_{1)}\left(r,\frac{1}{g'}\right) + S(r), \end{aligned} \tag{3.11}$$

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$$(3.12)\qquad \overline{N}\left(r,\frac{1}{f-1}\right) = N_{1)}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) \leq N_{1)}\left(r,\frac{1}{f'}\right) + N_{1)}\left(r,\frac{1}{g'}\right) + S(r),$$

and

$$(3.13)\qquad \overline{N}(r,f) = N_{1)}(r,f) + \overline{N}_{(2}(r,f) \leq N_{1)}\left(r,\frac{1}{f'}\right) + N_{1)}\left(r,\frac{1}{g'}\right) + S(r).$$

From (1.2), (3.10), (3.11), (3.12), (3.13), and the second fundamental theorem, we have for $r \in I$

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-1}\right) + S(r) \\ &\leq 3\left[N_{1)}\left(r,\frac{1}{f'}\right) + N_{1)}\left(r,\frac{1}{g'}\right)\right] + S(r) \\ &< 3(\lambda + o(1))T(r,f), \end{aligned}$$

which is impossible since $0 < \lambda < 1/3$. Therefore f is a fractional linear transformation of g. Again from Lemma 2.5, we obtain the conclusion of Theorem 1.5. □

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4. Final Remarks

Clearly, if k_j $(j = 1, 2, 3)$ are positive integers satisfying (1.1), then

$$k_j k_i > 1 \quad (j \neq i, j, i = 1, 2, 3).$$

Theorem 4.1. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_1 and k_2 are positive integers satisfying:*

$$k_1 k_2 > 1. \tag{4.1}$$

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f - 1)(g - 1) \equiv 1$,

(iv) $f + g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g - 1)$,

(vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ is a constant.

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Theorem 4.2. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k), (a_2, ∞), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k is an integer satisfying:*

$$k \geq 1. \tag{4.2}$$

If $\overline{E}_{1)}(0, f') \subseteq \overline{E}_{\infty)}(0, g')$ and $\overline{E}_{1)}(0, g') \subseteq \overline{E}_{\infty)}(0, f')$, then f and g satisfy one of the following relations:

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f - 1)(g - 1) \equiv 1$,

(iv) $f + g \equiv 1$,

(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g - 1)$,

(vii) $[(c - 1)f + 1][(c - 1)g - c] \equiv -c$,
where c $(\neq 0, 1)$ is a constant.

Theorem 4.3. *Let f and g be two nonconstant meromorphic functions sharing (a_1, k_1), (a_2, k_2), and (a_3, ∞), where $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, and k_1 and k_2 are positive integers satisfying (4.1). If (1.2) holds, then f and g satisfy one of the following relations:*

(i) $f \equiv g$,

(ii) $fg \equiv 1$,

(iii) $(f - 1)(g - 1) \equiv 1$,

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(iv) $f+g\equiv 1$,

(v) $f\equiv cg$,

(vi) $f-1\equiv c(g-1)$,

(vii) $[(c-1)f+1][(c-1)g-c]\equiv -c$,

where $c\ (\neq 0,1)$ *is a constant.*

Theorem 4.4. *Let f and g be two nonconstant meromorphic functions sharing (a_1,k), (a_2,∞), and (a_3,∞), where $\{a_1,a_2,a_3\}=\{0,1,\infty\}$, and k is an integer satisfying (4.2). If (1.2) holds, then f and g satisfy one of the following relations:*

(i) $f\equiv g$,

(ii) $fg\equiv 1$,

(iii) $(f-1)(g-1)\equiv 1$,

(iv) $f+g\equiv 1$,

(v) $f\equiv cg$,

(vi) $f-1\equiv c(g-1)$,

(vii) $[(c-1)f+1][(c-1)g-c]\equiv -c$,

where $c\ (\neq 0,1)$ *is a constant.*

Proofs of Theorems 4.1 and 4.3. Without loss of generality, we assume that $k_1\leq k_2$. Then by (4.1) we see that $k_1\geq 1$ and $k_2\geq 2$. Note that if f and g share (a,k) then f and g share (a,p) for all integers p, $0\leq p<k$. Since f and g share (a_1,k_1), (a_2,k_2), and (a_3,∞), it follows that f and g share $(a_1,1)$, $(a_2,2)$, and $(a_3,6)$. Thus form Corollaries 1.4 and 1.6 we immediately obtain the conclusions of Theorems 4.1 and 4.3 respectively. □

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Proofs of Theorems 4.2 and 4.4. Note that if f and g share (a_1, k), (a_2, ∞), (a_3, ∞), and $k \geq 1$, then we know that f and g share $(a_1, 1)$, $(a_2, 2)$, and $(a_3, 6)$. Thus from Corollaries 1.4 and 1.6 we instantly get the conclusions of Theorems 4.2 and 4.4 respectively. □

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