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A GENERALIZED RESIDUAL INFORMATION MEASURE AND ITS PROPERTIES

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ABSTRACT. Ebrahim and Pellery [7] and Ebrahim [4] proposed the Shannon residual entropy function as a dynamic measure of uncertainty. In this paper we introduce and study a generalized information measure for residual lifetime distributions. It is shown that the proposed measure uniquely determines the distribution function. Also, characterization results for some lifetime distributions are discussed. Some discrete distribution results are also addressed.

Key words and phrases: Shannon entropy, Renyi entropy, Residual entropy, Generalized residual entropy, Life time distributions.

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1. INTRODUCTION

Let X be an absolutely continuous non-negative variable describing the random lifetime of a component. Let $f(x)$ be the probability density function, $F(x)$ be the cumulative distribution and $R(x)$ be the survival function of the random variable X. A classical measure of uncertainty for X is the differential entropy, also known as the Shannon information measure, defined as

$$H(X) = -\int_0^\infty f(x)\log f(x)dx. \tag{1.1}$$

If X is a discrete random variable taking values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$, then Shannon's entropy is defined as

$$H(P) = H(p_1, p_2, ..., p_n) = -\sum_{k=1}^{n} p_k \log(p_k). \tag{1.2}$$

Renyi [11] generalized (1.1) and defined the measure

$$H_\alpha(X) = \frac{1}{\alpha(1-\alpha)}\log\int_0^\infty f^\alpha(x)dx, \quad \alpha > 1 \tag{1.3}$$

174-09

and in the discrete case

$$H_\alpha(X) = \frac{1}{\alpha(1-\alpha)} \log \sum_{k=1}^{n} p_k^\alpha, \quad \alpha > 1. \tag{1.4}$$

Furthermore, in the continous case

$$\lim_{\alpha\to 1} H_\alpha(X) = -\int_0^\infty f(x) \log f(x) dx = H(X) \tag{1.5}$$

and in discrete case

$$\lim_{\alpha\to 1} H_\alpha(X) = -\sum_{k=1}^{n} p_k \log(p_k) = H(P), \tag{1.6}$$

which is Shannon's entropy in both cases.

The role of differential entropy as a measure of uncertainty in residual lifetime distributions has attracted increasing attention in recent years. As stated by Ebrahimi [4], the residual entropy at a time t of a random life time X is defined as the differential entropy of $(X/X > t)$. Formally, for all $t > 0$, the residual entropy of X is given by

$$H(X;t) = -\int_t^\infty \frac{f(x)}{R(t)} \log \frac{f(x)}{R(t)} dx \tag{1.7}$$

or

$$H(X;t) = 1 - \frac{1}{R(t)} \int_t^\infty f(x) \log h(x) dx,$$

where $h(t) = \frac{f(t)}{R(t)}$ is the hazard function or failure rate of the random variable X. Given that an item has survived up to t, $H(X;t)$ measures the uncertainty of the remaining lifetime of the component.

In the case of a discrete random variable, we have

$$H(t_j) = -\sum_{k=j}^{n} \frac{P(t_k)}{R(t_j)} \log \frac{P(t_k)}{R(t_j)}, \tag{1.8}$$

where $R(t)$ is the reliability function of the random variable X.

Nair and Rajesh [9] studied the characterization of lifetime distributions by using the residual entropy function corresponding to the Shannon's entropy. In this sequel, we investigate the problem of the characterization of a lifetime distribution using the following generalized residual entropy function:

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)} \log \left(\frac{\int_t^\infty f^\alpha(x) dx}{R^\alpha(t)} \right), \quad \alpha > 1. \tag{1.9}$$

As $\alpha \to 1$, (1.9) reduces to (1.7).

The measure (1.9) is the residual life entropy corresponding to (1.3).

2. Characterization of Distributions

2.1. **Continuous Case.** Let X be a continuous non-negative random variable representing component failure time with failure distribution $F(t) = P(X \leq t)$ and survival function $R(t) = 1 - F(t)$ with $R(0) = 1$. We define the generalized entropy for residual life as

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)} \log \left(\frac{\int_t^\infty f^\alpha(x) dx}{R^\alpha(t)} \right), \quad \alpha > 1 \tag{2.1}$$

and so

$$\int_t^\infty f^\alpha(x)dx = R^\alpha(t)\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right), \quad \alpha > 1. \tag{2.2}$$

We now show that $H_\alpha(X;t)$ uniquely determines $R(t)$.

Theorem 2.1. *If X has an absolutely continuous distribution $F(t)$ with reliability function $R(t)$ and an increasing residual entropy $H_\alpha(X;t)$, then $H_\alpha(X;t)$ uniquely determines $R(t)$.*

Proof. Differentiating (2.2) with respect to t, we have

$$\begin{aligned} h^\alpha(t) &= \alpha h(t)\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right) \\ &\quad - (\alpha)(1-\alpha)\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right)H'_\alpha(X;t), \end{aligned} \tag{2.3}$$

where $h(t) = \frac{f(t)}{R(t)}$ is the failure rate function.

Hence for a fixed $t > 0$, $h(t)$ is a solution of

$$\begin{aligned} g(x) &= (x)^\alpha - \alpha x\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right) \\ &\quad + \alpha(1-\alpha)\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right)H'_\alpha(X;t) = 0. \end{aligned} \tag{2.4}$$

Differentiating both sides with respect to x, we have

$$g'(x) = \alpha(x)^{\alpha-1} - \alpha\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right). \tag{2.5}$$

Now for $\alpha > 1,\ g(0) \le 0,\ g(\infty) = \infty,\ g(x)$ first decreases and then increases with minimum at $x_t = \exp\left(-\alpha H_\alpha(X;t)\right)$.

So, the unique solution to $g(x) = 0$ is given by $x = h(t)$. Thus $H_\alpha(X;t)$ determines $h(t)$ uniquely and hence determines $R(t)$ uniquely. □

Theorem 2.2. *The uniform distribution over $(a,b),\ a < b$ can be characterized by a decreasing generalized residual entropy $H_\alpha(X;t) = \frac{1}{\alpha}\log(b-t),\ b > t$.*

Proof. For the case of uniform distribution over $(a,b),\ a < b$, we have

$$H_\alpha(X;t) = \frac{1}{\alpha}\log(b-t), \qquad b > t \tag{2.6}$$

which is decreasing in t.

Also, $x_t = \exp\left(-\alpha H_\alpha(X;t)\right)$, therefore,

$$\begin{aligned} g(x_t) &= (x_t)^\alpha - \alpha x_t\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right) + \alpha(1-\alpha)\exp\left(\alpha(1-\alpha)H_\alpha(X;t)\right)H'_\alpha(X;t) \\ &= 0. \end{aligned}$$

Hence $H_\alpha(X;t) = \frac{1}{\alpha}\log(b-t)$ is the unique solution to $g(x_t) = 0$, which proves the theorem. □

Theorem 2.3. *Let X be a random variable having a generalized residual entropy of the form*

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)}\log k - \frac{1}{\alpha}\log h(t), \tag{2.7}$$

where $h(t)$ is the failure rate function of X. Then X has

(i) *an exponential distribution iff $k = \frac{1}{\alpha}$,*
(ii) *a Pareto distribution iff $k < \frac{1}{\alpha}$ and*
(iii) *a finite range distribution iff $k > \frac{1}{\alpha}$.*

Proof. (i) Let X have the exponential distribution,

$$f(t) = \frac{1}{\theta} \exp\left(-\left(\frac{t}{\theta}\right)\right), \quad t > 0,\ \theta > 0.$$

The reliability function is given by

$$R(t) = \exp\left(-\frac{t}{\theta}\right)$$

and the failure rate function by

$$h(t) = \frac{1}{\theta}.$$

Therefore, after simplification, using (2.1),

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)} \log k - \frac{1}{\alpha} \log h(t), \tag{2.8}$$

where $k = \frac{1}{\alpha}$ and $h(t) = \frac{1}{\theta}$.

Thus (2.7) holds.

Conversely, suppose that $k = \frac{1}{\alpha}$, then

$$\frac{1}{\alpha(1-\alpha)} \log k - \frac{1}{\alpha} \log h(t) = \frac{1}{\alpha(1-\alpha)} \log\left(\frac{\int_t^\infty f^\alpha(x)dx}{R^\alpha(t)}\right), \quad \alpha > 1$$

which gives,

$$h(t) = \left(\frac{1-k\alpha}{k(\alpha-1)} t + \frac{1}{h(0)}\right)^{-1} = (at+b)^{-1}, \tag{2.9}$$

where $a = \left(\frac{1-k\alpha}{k(\alpha-1)}\right) = 0$, since $k = \frac{1}{\alpha}$ and $b = \frac{1}{h(0)}$.

Clearly (2.9) is the failure rate function of the exponential distribution.

(ii) The density function of the Pareto distribution is given by

$$f(t) = \frac{(b)^{\frac{1}{a}}}{(at+b)^{1+\frac{1}{a}}}, \quad t \geq 0,\ a > 0,\ b > 0.$$

The reliability function is given by

$$R(t) = \frac{(b)^{\frac{1}{a}}}{(at+b)^{\frac{1}{a}}}, \quad t \geq 0,\ a > 0,\ b > 0$$

and failure rate is given by

$$h(t) = (at+b)^{-1}. \tag{2.10}$$

After simplification, (2.1) yields

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)} \log k - \frac{1}{\alpha} \log h(t), \tag{2.11}$$

where $k = \frac{1}{a\alpha+\alpha-a} < \frac{1}{\alpha}$, since $\alpha > 1$ and $h(t) = (at+b)^{-1}$.

Thus (2.7) holds.

Conversely, suppose that $k < \frac{1}{\alpha}$. Proceeding as in (i), (2.9) gives

$$h(t) = \left(\frac{1-k\alpha}{k(\alpha-1)} t + \frac{1}{h(0)}\right)^{-1} = (at+b)^{-1}, \tag{2.12}$$

where $a = \left(\frac{1-k\alpha}{k(\alpha-1)}\right) > 0$, since $k < \frac{1}{\alpha},\ \alpha > 1$ and $b = \frac{1}{h(0)}$.

Clearly, (2.12) is the failure rate function of the Pareto distribution given in (2.10).
(iii) The density function of the finite range distribution is given by

$$f(t) = \frac{\beta_1}{\nu}\left(1-\frac{t}{\nu}\right)^{\beta_1-1}, \quad \beta_1 > 0, 0 \le t \le \nu < \infty.$$

The reliability function is given by

$$R(t) = \left(1-\frac{t}{\nu}\right)^{\beta_1}, \quad \beta_1 > 0, 0 \le t \le \nu < \infty$$

and the failure rate function by

$$h(t) = \left(\frac{\beta_1}{\nu}\right)\left(1-\frac{t}{\nu}\right)^{-1}. \tag{2.13}$$

It follows that

$$H_\alpha(X;t) = \frac{1}{\alpha(1-\alpha)}\log k - \frac{1}{\alpha}\log h(t),$$

where $k = \frac{\beta_1}{\alpha\beta_1-\alpha+1} > \frac{1}{\alpha}$, since $\alpha > 1$ and $h(t) = \left(\frac{\beta_1}{\nu}\right)\left(1-\frac{t}{\nu}\right)^{-1}$.

Thus (2.7) holds.

Conversely, suppose $k > \frac{1}{\alpha}$. Proceeding as in (i), (2.9) gives

$$h(t) = h(0)\left(1-\frac{k\alpha-1}{k(\alpha-1)}h(0)t\right)^{-1}, \tag{2.14}$$

which is the failure rate function of the distribution given by (2.13), if $k > \frac{1}{\alpha}$. □

2.2. **Discrete Case.** Let X be a discrete random variable taking values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$. The discrete residual entropy is defined as

$$H(p;j) = -\sum_{k=j}^{n}\frac{p_k}{R(j)}\log\left(\frac{p_k}{R(j)}\right). \tag{2.15}$$

The generalized residual entropy for the discrete case is defined as

$$H_\alpha(p;j) = \frac{1}{\alpha(1-\alpha)}\log\sum_{k=j}^{n}\left(\frac{p_k}{R(j)}\right)^\alpha. \tag{2.16}$$

For $\alpha \to 1$, (2.16) reduces to (2.15).

Theorem 2.4. *If X has a discrete distribution $F(t)$ with support $(t_j : t_j < t_{j+1})$ and an increasing generalized residual entropy $H_\alpha(X;t)$ then $H_\alpha(X;t)$ uniquely determines $F(t)$.*

Proof. We have

$$H_\alpha(p;j) = \frac{1}{\alpha(1-\alpha)}\log\sum_{k=j}^{n}\left(\frac{p_k}{R(j)}\right)^\alpha$$

and so

$$\sum_{k=j}^{n} p_k^\alpha = R^\alpha(j)\exp\left(\alpha(1-\alpha)H_\alpha(p;j)\right). \tag{2.17}$$

For $j+1$, we have

$$\sum_{k=j+1}^{n} p_k^\alpha = R^\alpha(j+1)\exp\left(\alpha(1-\alpha)H_\alpha(p;j+1)\right). \tag{2.18}$$

Subtracting (2.18) from (2.17), using $p_j = R(j) - R(j+1)$ and $\lambda_j = \frac{R(j+1)}{R(j)}$, we have

$$\exp\left(\alpha(1-\alpha)H_\alpha(p;j)\right) = (1-\lambda_j)^\alpha + (\lambda_j)^\alpha \exp\left(\alpha(1-\alpha)H_\alpha(p;j+1)\right).$$

Hence, λ_j is a number in $(0,1)$ which is a solution of

$$\phi(x) = \exp\left(\alpha(1-\alpha)H_\alpha(p;j)\right) - (1-x)^\alpha - (x)^\alpha \exp\left(\alpha(1-\alpha)H_\alpha(p;j+1)\right). \tag{2.19}$$

Differentiating both sides with respect to x, we have

$$\phi'(x) = \alpha(1-x)^{\alpha-1} - \alpha(x)^{\alpha-1} \exp\left(\alpha(1-\alpha)H_\alpha(p;j+1)\right). \tag{2.20}$$

Note that $\phi'(x) = 0$ gives

$$x = \left[1 + \exp\left(-\alpha H_\alpha(p;j+1)\right)\right]^{-1} = x_j.$$

Now for $\alpha > 1$, $\phi(0) \leq 0$ and $\phi(1) \leq 0$, $\phi(x)$ first increases and then decreases in $(0,1)$ with a maximum at $x_j = \left[1 + \exp\left(-\alpha H_\alpha(p;j+1)\right)\right]^{-1}$.

So the unique solution to $\phi(x) = 0$ is given by $x = x_j$.

Thus $H_\alpha(X;t)$ uniquely determines $F(t)$. □

Theorem 2.5. *A discrete uniform distribution with support* $(1,2,...,n)$ *is characterized by the decreasing generalized discrete residual entropy*

$$H_\alpha(p;j) = \frac{1}{\alpha}\log(n-j+1), \quad j = 1,2,...,n.$$

Proof. In the case of a discrete uniform distribution with support $(1,2,...,n)$,

$$H_\alpha(p;j) = \frac{1}{\alpha}\log(n-j+1), \quad j = 1,2,...,n$$

which is decreasing in j.

Also,

$$x_j = \left[1 + \exp\left(-\alpha H_\alpha(p;j+1)\right)\right]^{-1}.$$

Therefore,

$$\begin{aligned}\phi(x_j) &= \exp\left(\alpha(1-\alpha)H_\alpha(p;j)\right) - (1-x_j)^\alpha - (x_j)^\alpha \exp\left(\alpha(1-\alpha)H_\alpha(p;j+1)\right)\\ &= 0\end{aligned}$$

which proves the theorem. □

3. A New Class of Life Time Distribution

Ebrahimi [4] defined two nonparametric classes of distribution based on the measure $H(X;t)$ as follows:

Definition 3.1. A random variable X is said to have decreasing (increasing) uncertainty in residual life DURL (IURL) if $H(X;t)$ is decreasing (increasing) in $t \geq 0$.

Definition 3.2. A non-negative random variable X is said to have decreasing (increasing) uncertainty in a generalized residual entropy of order α DUGRL(IUGRL) if $H_\alpha(X;t)$ is decreasing (increasing) in t, $t > 0$.

This implies that the random variable X has DUGRL(IUGRL),

$$H'_\alpha(X;t) \leq 0,$$
$$H'_\alpha(X;t) \geq 0.$$

Now we present a relationship between the new classes and the decreasing(increasing) failure rate class of lifetime distributions.

Remark 1. R is said to be an IFR(DFR) if $h(t)$ is increasing(decreasing) in t.

Theorem 3.1. *If R has an increasing(decreasing) failure rate, IFR(DFR) then it is also a DUGRL(IUGRL).*

Proof. We have,

$$H'_\alpha(X;t) = \frac{1}{1-\alpha}\left[h(t) - h^\alpha(t)\exp\left(-\alpha(1-\alpha)H_\alpha(X;t)\right)\right]. \tag{3.1}$$

Since R is IFR, by (3.1) and Remark 1, we have

$$H'_\alpha(X;t) \leq 0,$$

which means that $H_\alpha(X;t)$ is decreasing in t, i.e, R is DUGRL. The proof for IUGRL is similar. □

Theorem 3.2. *If a distribution is DUGRL as well as IUGRL for some constant, then it must be exponential.*

Proof. Since the random variable X is both DUGRL and IUGRL, then,

$$H_\alpha(X;t) = \text{constant}.$$

Differentiating both sides with respect to t, we get

$$h(t) = \text{constant},$$

which means that the distribution is exponential. □

The following lemma which gives the value of the function $H_\alpha(X;t)$ under linear transformation will be used in proving the upcoming theorem.

Lemma 3.3. *For any absolutely continuous random variable X, define $Z = ax + b$, where $a > 0, b \geq 0$ are constants, then*

$$H_\alpha(Z;t) = \frac{\log a}{\alpha} + H_\alpha\left(X;\frac{t-b}{a}\right).$$

Proof. We have, $H_\alpha(X;t)$ from (2.1) and $Z = ax + b$, therefore,

$$H_\alpha(Z;t) = \frac{\log a}{\alpha} + H_\alpha\left(X;\frac{t-b}{a}\right),$$

which proves the lemma. □

Theorem 3.4. *Let X be an absolutely continuous random variable and $X \in DUGRL(IUGRL)$. Define $Z = aX + b$, where $a > 0$ and $b \geq 0$ are constants, then $Z \in DUGRL(IUGRL)$.*

Proof. Since $X \in DUGRL(IUGRL)$, then,

$$H'_\alpha(X;t) \leq 0,$$
$$H'_\alpha(X;t) \geq 0.$$

By applying Lemma 3.3, it follows that $Z \in DUGRL(IUGRL)$, which proves the theorem. □

The next theorem gives upper(lower) bounds for the failure rate function.

Theorem 3.5. *If X is DUGRL(IUGRL), then*

$$h(t) \geq (\leq)(\alpha)^{\frac{1}{\alpha-1}}\exp\left(-\alpha H_\alpha(X;t)\right).$$

Proof. If X is DUGRL, then

$$H'_\alpha(X;t) \leq 0$$

which gives,

$$h(t) \geq (\alpha)^{\frac{1}{\alpha-1}} \exp\left(-\alpha H_\alpha(X;t)\right). \tag{3.2}$$

Similarly, if X is IUGRL, then

$$h(t) \leq (\alpha)^{\frac{1}{\alpha-1}} \exp\left(-\alpha H_\alpha(X;t)\right). \tag{3.3}$$

□

Corollary 3.6. *Let $R(t)$ be a DUGRL(IUGRL), then*

$$R(t) \leq (\geq) \exp\left(-\int_0^t (\alpha)^{\frac{1}{\alpha-1}} \exp\left(-\alpha H_\alpha(X;u)du\right)\right)$$

for all $t \geq 0$.

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