

# Journal of Inequalities in Pure and Applied Mathematics

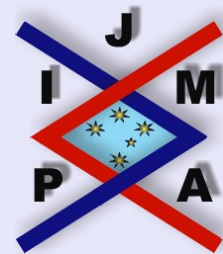
## MARKOFF-TYPE INEQUALITIES IN WEIGHTED $L^2$ -NORMS

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Abstract

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## Abstract

We give exact estimations of certain weighted  $L^2$ -norms of the  $k$ -th derivative of polynomials which have a curved majorant. They are all obtained as applications of special quadrature formulae.

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*Key words:* Bouzitat quadrature, Chebyshev polynomials, Inequalities.

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# 1. Introduction

The following problem was raised by P.Turán:

Let  $\varphi(x) \geq 0$  for  $-1 \leq x \leq 1$  and consider the class  $P_{n,\varphi}$  of all polynomials of degree  $n$  such that  $|p_n(x)| \leq \varphi(x)$  for  $-1 \leq x \leq 1$ . How large can  $\max_{[-1,1]} |p_n^{(k)}(x)|$  be if  $p_n$  is an arbitrary polynomial in  $P_{n,\varphi}$ ?

The aim of this paper is to consider the solution in the weighted  $L^2$ -norm for the majorant  $\varphi(x) = \frac{\alpha + \beta - 2\alpha x^2}{\sqrt{1-x^2}}$ ,  $0 \leq \alpha \leq \beta$ .

Let us denote by

$$(1.1) \quad x_i = \cos \frac{(2i-1)\pi}{2n}, \quad i = 1, 2, \dots, n,$$

the zeros of  $T_n(x) = \cos n\theta, x = \cos \theta$ ,

the Chebyshev polynomial of the first kind,

$$(1.2) \quad y_i^{(k)} \text{ the zeros of } U_{n-1}^{(k)}(x), \quad U_{n-1}(x) = \frac{\sin n\theta}{\sin \theta}, \quad x = \cos \theta,$$

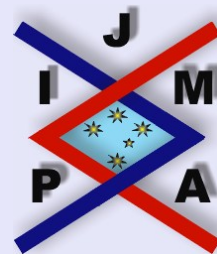
the Chebyshev polynomial of the second kind and

$$(1.3) \quad G_{n-1}(x) = \beta U_{n-1}(x) - \alpha U_{n-3}(x), \quad 0 \leq \alpha \leq \beta.$$

Let  $H_{\alpha,\beta}$  be the class of all real polynomials  $p_{n-1}$ , of degree  $\leq n-1$  such that

$$(1.4) \quad |p_{n-1}(x_i)| \leq \frac{\alpha + \beta - 2\alpha x_i^2}{\sqrt{1-x_i^2}}, \quad i = 1, 2, \dots, n,$$

where the  $x_i$ 's are given by (1.1) and  $0 \leq \alpha \leq \beta$ .



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## 2. Results

**Theorem 2.1.** *If  $p_{n-1} \in H_{\alpha,\beta}$  then we have*

$$(2.1) \quad \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [p'_{n-1}(x)]^2 dx \leq \frac{2\pi(n-1)}{15} [(\alpha-\beta)^2 n(n+1)(n-2)(n-3) + 5(n-1)(\beta^2 n(n+1) - \alpha^2(n-2)(n-3))]$$

*with equality for  $p_{n-1} = G_{n-1}$ .*

Two cases are of special interest:

**I. Case**  $\alpha = \beta = \frac{1}{2}$ ,  $\varphi(x) = \sqrt{1-x^2}$  (**circular majorant**),  $G_{n-1} = T_{n-1}$ .

Note that  $P_{n-1,\varphi} \subset H_{\frac{1}{2},\frac{1}{2}}$ ,  $T_{n-1} \notin P_{n-1,\varphi}$ ,  $T_{n-1} \in H_{\frac{1}{2},\frac{1}{2}}$ .

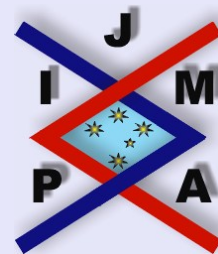
**Corollary 2.2.** *If  $p_{n-1} \in H_{\frac{1}{2},\frac{1}{2}}$  then we have*

$$(2.2) \quad \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [p'_{n-1}(x)]^2 dx \leq \pi(n-1)^3,$$

*with equality for  $p_{n-1} = T_{n-1}$ .*

**II. Case**  $\alpha = 0$ ,  $\beta = 1$ ,  $\varphi(x) = \frac{1}{\sqrt{1-x^2}}$ ,  $G_{n-1} = U_{n-1}$ .

Note that  $P_{n-1,\varphi} \subset H_{0,1}$ ,  $U_{n-1} \in P_{n-1,\varphi}$ ,  $U_{n-1} \in H_{0,1}$ .



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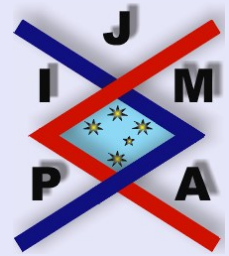


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**Corollary 2.3.** *If  $p_{n-1} \in H_{0,1}$  then we have*

$$(2.3) \quad \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [p'_{n-1}(x)]^2 dx \leq \frac{2\pi n(n^4-1)}{15},$$

*with equality for  $p_{n-1} = U_{n-1}$ .*

In this second case we have a more general result:

**Theorem 2.4.** *If  $p_{n-1} \in H_{0,1}$  and*

$$r(x) = b(b-2a)x^2 + 2c(b-a)x + a^2 + c^2$$

*with  $0 < a < b, |c| < b - a, b \neq 2a$  then we have*

$$(2.4) \quad \int_{-1}^1 r(x) (1-x^2)^{k-1/2} [p_{n-1}^{(k+1)}(x)]^2 dx \leq \frac{\pi(n+k+1)!}{(n-k-2)!} \\ \times \left[ \frac{[2(n^2-k^2) - 3(2k+1)][(a-b)^2 + c^2]}{(2k+1)(2k+3)(2k+5)} + \frac{a^2 + c^2}{2k+3} \right],$$

*where  $k = 0, \dots, n-2$ , with equality for  $p_{n-1} = U_{n-1}$ .*

Setting  $a = 1, b = c = 0$  one obtains the following

**Corollary 2.5.** *If  $p_{n-1} \in H_{0,1}$  then we have*

$$(2.5) \quad \int_{-1}^1 (1-x^2)^{k-1/2} [p_{n-1}^{(k+1)}(x)]^2 dx \\ \leq \frac{2\pi(n+k+1)!}{(n-k-2)!} \cdot \frac{n^2 + k^2 + 3k + 1}{(2k+1)(2k+3)(2k+5)},$$

*$k = 0, \dots, n-2$ , with equality for  $p_{n-1} = U_{n-1}$ .*

### 3. Lemmas

Here we state and prove some lemmas which help us in proving the above theorems.

**Lemma 3.1.** *Let  $p_{n-1}$  be such that  $|p_{n-1}(x_i)| \leq \frac{\alpha+\beta-2\alpha x_i^2}{\sqrt{1-x_i^2}}, i = 1, 2, \dots, n$ , where the  $x_i$ 's are given by (1.1). Then we have*

$$(3.1) \quad |p'_{n-1}(y_j)| \leq |G'_{n-1}(y_j)|, \quad k = 0, 1, \dots, n-1,$$

and

$$(3.2) \quad |p'_{n-1}(1)| \leq |G'_{n-1}(1)|, \quad |p'_{n-1}(-1)| \leq |G'_{n-1}(-1)|.$$

*Proof.* By the Lagrange interpolation formula based on the zeros of  $T_n$  and using  $T'_n(x_i) = \frac{(-1)^{i+1}n}{(1-x_i^2)^{1/2}}$ , we can represent any algebraic polynomial  $p_{n-1}$  by

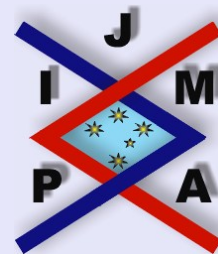
$$p_{n-1}(x) = \frac{1}{n} \sum_{i=1}^n \frac{T_n(x)}{x-x_i} (-1)^{i+1} (1-x_i^2)^{1/2} p_{n-1}(x_i).$$

From

$$G_{n-1}(x_i) = (-1)^{i+1} \frac{\alpha + \beta - 2\alpha x_i^2}{\sqrt{1-x_i^2}}$$

we have

$$G_{n-1}(x) = \frac{1}{n} \sum_{i=1}^n \frac{T_n(x)}{x-x_i} (\alpha + \beta - 2\alpha x_i^2).$$



Differentiating with respect to  $x$  we obtain

$$p'_{n-1}(x) = \frac{1}{n} \sum_{i=1}^n \frac{T'_n(x)(x-x_i) - T_n(x)}{(x-x_i)^2} (-1)^{i+1} (1-x_i^2)^{1/2} p_{n-1}(x_i).$$

On the roots of  $T'_n(x) = nU_{n-1}(x)$  and using (1.4) we find

$$\begin{aligned} |p'_{n-1}(y_j)| &\leq \frac{1}{n} \sum_{i=1}^n \frac{|T_n(y_j)|}{(y_j-x_i)^2} (\alpha + \beta - 2\alpha x_i^2) \\ &= \frac{|T_n(y_j)|}{n} \sum_{i=1}^n \frac{\alpha + \beta - 2\alpha x_i^2}{(y_j-x_i)^2} = |G'_{n-1}(y_j)|. \end{aligned}$$

For  $l_i(x) = \frac{T_n(x)}{x-x_i}$  taking into account that  $l'_i(1) > 0$  (see [6]) it follows that

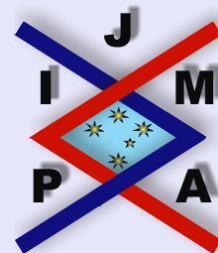
$$|p'_{n-1}(1)| \leq \frac{1}{n} \sum_{i=1}^n l'_i(1) (\alpha + \beta - 2\alpha x_i^2) = |G'_{n-1}(1)|.$$

Similarly  $|p'_{n-1}(-1)| \leq |G'_{n-1}(-1)|$ . □

We shall need the result of Duffin and Schaeffer [2]:

**Lemma 3.2 (Duffin – Schaeffer).** *If  $q(x) = c \prod_{i=1}^n (x-x_i)$  is a polynomial of degree  $n$  with  $n$  distinct real zeros and if  $p \in P_n$  is such that*

$$|p'(x_i)| \leq |q'(x_i)| \quad (i = 1, 2, \dots, n),$$



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then for  $k = 1, 2, \dots, n - 1$ ,

$$|p^{(k+1)}(x)| \leq |q^{(k+1)}(x)|$$

whenever  $q^{(k)}(x) = 0$ .

**Lemma 3.3.** Let  $p_{n-1}$  be such that  $|p_{n-1}(x_i)| \leq \frac{1}{\sqrt{1-x_i^2}}$ ,  $i = 1, 2, \dots, n$ , where the  $x_i$ 's are given by (1.1). Then we have

$$(3.3) \quad |p_{n-1}^{(k+1)}(y_j^{(k)})| \leq |U_{n-1}^{(k+1)}(y_j^{(k)})|, \quad \text{whenever } U_{n-1}^{(k)}(y_j^{(k)}) = 0,$$

for  $k = 0, 1, \dots, n - 1$ , and

$$(3.4) \quad |p_{n-1}^{(k+1)}(1)| \leq |U_{n-1}^{(k+1)}(1)|, \quad |p_{n-1}^{(k+1)}(-1)| \leq |U_{n-1}^{(k+1)}(-1)|.$$

*Proof.* For  $\alpha = 0, \beta = 1, G_{n-1} = U_{n-1}$  and (3.1) give  $|p'_{n-1}(y_j)| \leq |U'_{n-1}(y_j)|$  and (3.2)

$$|p'_{n-1}(1)| \leq |U'_{n-1}(1)|, \quad |p'_{n-1}(-1)| \leq |U'_{n-1}(-1)|.$$

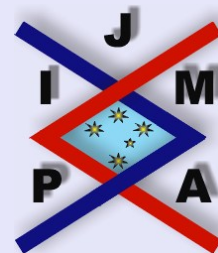
Now the proof is concluded by applying the Duffin-Schaeffer lemma.  $\square$

The following proposition was proved in [3].

**Lemma 3.4.** A real polynomial  $r$  of exact degree 2 satisfies  $r(x) > 0$  for  $-1 \leq x \leq 1$  if and only if

$$r(x) = b(b - 2a)x^2 + 2c(b - a)x + a^2 + c^2$$

with  $0 < a < b$ ,  $|c| < b - a$ ,  $b \neq 2a$ .



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We need the following quadrature formulae:

**Lemma 3.5.** For any given  $n$  and  $k$ ,  $0 \leq k \leq n-1$ , let  $y_i^{(k)}$ ,  $i = 1, \dots, n-k-1$ , be the zeros of  $U_{n-1}^{(k)}$ . Then the quadrature formulae

$$(3.5) \quad \int_{-1}^1 (1-x^2)^{k-1/2} f(x) dx = A_0 [f(-1) + f(1)] + \sum_{i=1}^{n-k-1} s_i f(y_i^{(k)}),$$

where

$$A_0 = \frac{2^{2k-1} (2k+1) \Gamma(k+1/2)^2 (n-k-1)!}{(n+k)!}, \quad s_i > 0$$

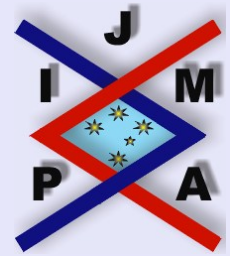
and

$$(3.6) \quad \int_{-1}^1 (1-x^2)^{k-1/2} f(x) dx = B_0 [f(-1) + f(1)] + C_0 [f'(-1) - f'(1)] + \sum_{i=1}^{n-k-2} v_i f(y_i^{(k+1)}),$$

where

$$C_0 = \frac{2^{2k} (2k+3) \Gamma(k+3/2)^2 (n-k-2)!}{(n+k+1)!},$$

$$B_0 = C_0 \frac{2(n^2 - (k+2)^2)(2k+3) + 4(k+1)(2k+5)}{(2k+1)(2k+5)}$$



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have algebraic degree of precision  $2n - 2k - 1$ .

The quadrature formulae

$$(3.7) \quad \int_{-1}^1 r(x) (1-x^2)^{k-1/2} f(x) dx = A_1 f(-1) + B_1 f(1) + \sum_{i=1}^{n-k-1} s_i r(y_i^{(k)}) f(y_i^{(k)}),$$

where

$$A_1 = \frac{2^{2k-1} (2k+1) \Gamma(k+1/2)^2 (n-k-1)! (a-b+c)^2}{(n+k)!},$$

$$B_1 = \frac{2^{2k-1} (2k+1) \Gamma(k+1/2)^2 (n-k-1)! (a-b-c)^2}{(n+k)!}$$

and

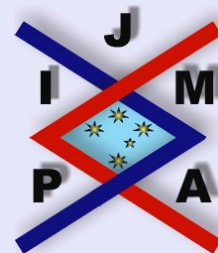
$$(3.8) \quad \int_{-1}^1 r(x) (1-x^2)^{k-1/2} f(x) dx = C_1 f(-1) + D_1 f(1) + C_2 f'(-1) - D_2 f'(1) + \sum_{i=1}^{n-k-2} v_i r(y_i^{(k+1)}) f(y_i^{(k+1)}),$$

$$C_1 = B_0 (a-b+c)^2 + 2C_0 d, D_1 = B_0 (a-b-c)^2 - 2C_0 e,$$

$$C_2 = C_0 (a-b+c)^2, D_2 = C_0 (a-b-c)^2,$$

$$d = 2ab + bc - ac - b^2, e = b^2 - 2ab + bc - ac.$$

have algebraic degree of precision  $2n - 2k - 3$ .



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*Proof.* In order to compute the coefficients we need the following formulae

$$(3.9) \quad \int_{-1}^1 (1-x)^\alpha (1+x)^\lambda P_m^{(\alpha,\beta)}(x) dx = \frac{(-1)^m 2^{\alpha+\lambda+1} \Gamma(\lambda+1) \Gamma(m+\alpha+1) \Gamma(\beta-\lambda+m)}{\Gamma(m+1) \Gamma(\beta-\lambda) \Gamma(m+\alpha+\lambda+2)}, \quad \lambda < \beta.$$

$$\int_{-1}^1 (1-x)^\lambda (1+x)^\beta P_m^{(\alpha,\beta)}(x) dx = \frac{2^{\beta+\lambda+1} \Gamma(\lambda+1) \Gamma(m+\beta+1) \Gamma(\alpha-\lambda+m)}{\Gamma(m+1) \Gamma(\alpha-\lambda) \Gamma(m+\beta+\lambda+2)}, \quad \lambda < \alpha$$

The first quadrature formula (3.5) is the Bouzitat formula of the second kind [4, formula (4.8.1)], for the zeros of  $U_{n-1}^{(k)} = cP_{n-k-1}^{(k+\frac{1}{2}, k+\frac{1}{2})}$ . Setting  $\alpha = \beta = \frac{1}{2}$ ,  $m = n - k - 1$  in [4, formula (4.8.5)] we find  $A_0$  and  $s_i > 0$  (cf. [4, formula (4.8.4)]).

If in the above quadrature formula (3.6), taking into account (3.9), we put

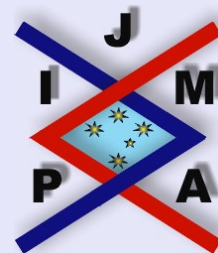
$$f(x) = (1-x)(1+x)^2 P_{n-k-2}^{(k+\frac{3}{2}, k+\frac{3}{2})}(x),$$

$$U_{n-1}^{(k+1)}(x) = cP_{n-k-2}^{(k+\frac{3}{2}, k+\frac{3}{2})}(x),$$

we obtain  $C_0$ , and for

$$f(x) = (1+x)^2 P_{n-k-2}^{(k+\frac{3}{2}, k+\frac{3}{2})}(x)$$

we find  $B_0$ .



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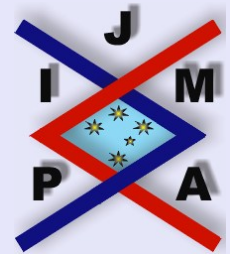
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If in formula (3.5) we replace  $f(x)$  with  $r(x)f(x)$  we get (3.7) and if in formula (3.6) we replace  $f(x)$  with  $r(x)f(x)$  we get (3.8).  $\square$



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## 4. Proof of the Theorems

*Proof of Theorem 2.1.* Setting  $k = 0$  in (3.5) we find the formula

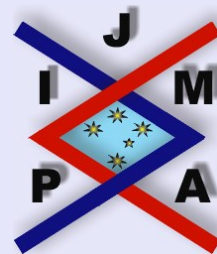
$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{2n} [f(-1) + f(1)] + \frac{\pi}{n} \sum_{i=1}^{n-1} f(y_i).$$

According to this quadrature formula and using (3.1) and (3.2) we have

$$\begin{aligned} & \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [p'_{n-1}(x)]^2 dx \\ &= \frac{\pi}{2n} (p'_{n-1}(-1))^2 + \frac{\pi}{2n} (p'_{n-1}(1))^2 + \frac{\pi}{n} \sum_{i=1}^{n-1} (p'_{n-1}(y_i))^2 \\ &\leq \frac{\pi}{2n} (G'_{n-1}(-1))^2 + \frac{\pi}{2n} (G'_{n-1}(1))^2 + \frac{\pi}{n} \sum_{i=1}^{n-1} (G'_{n-1}(y_i))^2 \\ &= \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} [G'_{n-1}(x)]^2 dx. \end{aligned}$$

Now

$$\begin{aligned} \int_{-1}^1 \frac{[G'_{n-1}(x)]^2}{\sqrt{1-x^2}} dx &= \beta^2 \int_{-1}^1 \frac{[U'_{n-1}(x)]^2}{\sqrt{1-x^2}} dx \\ &\quad - 2\alpha\beta \int_{-1}^1 \frac{U'_{n-1}(x)U'_{n-3}(x)}{\sqrt{1-x^2}} dx + \alpha^2 \int_{-1}^1 \frac{[U'_{n-3}(x)]^2}{\sqrt{1-x^2}} dx. \end{aligned}$$



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Using the following formula ( $k = 0$  in (3.6))

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{3\pi(3n^2-2)}{10n(n^2-1)} [f(-1) + f(1)] + \frac{3\pi}{4n(n^2-1)} [f'(-1) - f'(1)] + \sum_{i=1}^{n-2} c_i f(y_i)$$

we find

$$\int_{-1}^1 \frac{[U'_{n-1}(x)]^2}{\sqrt{1-x^2}} = \frac{2\pi n(n^4-1)}{15},$$

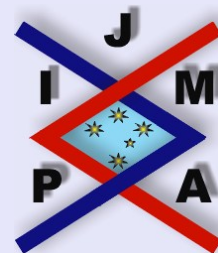
$$\int_{-1}^1 \frac{U'_{n-1}(x)U'_{n-3}(x)}{\sqrt{1-x^2}} = \frac{2\pi n(n^2-1)(n-2)(n-3)}{15},$$

$$\int_{-1}^1 \frac{[U'_{n-3}(x)]^2}{\sqrt{1-x^2}} = \frac{2\pi(n-1)(n^2-4n+5)(n-2)(n-3)}{15}$$

and

$$\int_{-1}^1 \frac{[G'_{n-1}(x)]^2}{\sqrt{1-x^2}} dx = \frac{2\pi(n-1)}{15} [(\alpha-\beta)^2 n(n+1)(n-2)(n-3) + 5(n-1)(\beta^2 n(n+1) - \alpha^2(n-2)(n-3))].$$

□



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*Proof of Theorem 2.4.* According to the quadrature formula (3.7), positivity of  $s_i$ 's, and using (3.3) and (3.4) we have

$$\begin{aligned} & \int_{-1}^1 r(x) (1-x^2)^{k-1/2} \left[ p_{n-1}^{(k+1)}(x) \right]^2 dx \\ &= A_1 \left[ p_{n-1}^{(k+1)}(-1) \right]^2 + B_1 \left[ p_{n-1}^{(k+1)}(1) \right]^2 + \sum_{i=1}^{n-k-1} s_i r(y_i^{(k)}) \left[ p_{n-1}^{(k+1)}(y_i^{(k)}) \right]^2 \\ &\leq A_1 \left[ U_{n-1}^{(k+1)}(-1) \right]^2 + B_1 \left[ U_{n-1}^{(k+1)}(1) \right]^2 + \sum_{i=1}^{n-k-1} s_i r(y_i^{(k)}) \left[ U_{n-1}^{(k+1)}(y_i^{(k)}) \right]^2 \\ &= \int_{-1}^1 r(x) (1-x^2)^{k-1/2} \left[ U_{n-1}^{(k+1)}(x) \right]^2 dx. \end{aligned}$$

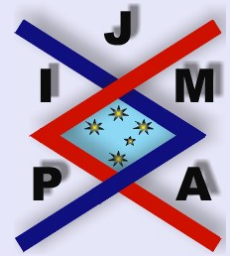
In order to complete the proof we apply formula (3.8) to  $f = \left[ U_{n-1}^{(k+1)}(x) \right]^2$ .

Having in mind  $U_{n-1}^{(k+1)}(y_i^{(k+1)}) = 0$  and the following relations deduced from [1]

$$U_{n-1}^{(k+1)}(1) = \frac{n(n^2-1^2) \cdots (n^2-(k+1)^2)}{1 \cdot 3 \cdots (2k+3)},$$

$$U_{n-1}^{(k+2)}(1) = \frac{n^2-(k+2)^2}{2k+5} U_{n-1}^{(k+1)}(1),$$

$$U_{n-1}^{(k+1)}(-1) U_{n-1}^{(k+2)}(-1) = -U_{n-1}^{(k+1)}(1) U_{n-1}^{(k+2)}(1),$$



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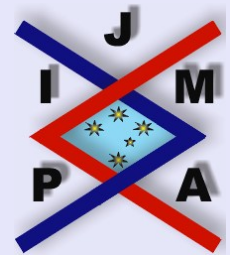
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we find

$$\begin{aligned}
 & \int_{-1}^1 r(x) (1-x^2)^{k-1/2} \left[ p_{n-1}^{(k+1)}(x) \right]^2 dx \\
 &= C_1 \left[ U_{n-1}^{(k+1)}(-1) \right]^2 + D_1 \left[ U_{n-1}^{(k+1)}(1) \right]^2 \\
 &\quad + 2C_2 U_{n-1}^{(k+1)}(-1) U_{n-1}^{(k+2)}(-1) - 2D_2 U_{n-1}^{(k+1)}(1) U_{n-1}^{(k+2)}(1) \\
 &= \frac{\pi(n+k+1)!}{(n-k-2)!} \left[ \frac{[2(n^2-k^2)-3(2k+1)][(a-b)^2+c^2]}{(2k+1)(2k+3)(2k+5)} + \frac{a^2+c^2}{2k+3} \right].
 \end{aligned}$$

□




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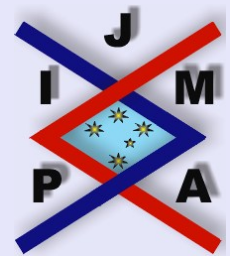
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