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ON SOME q-INTEGRAL INEQUALITIES

KAMEL BRAHIM

INSTITUT PRÉPARATOIRE AUX ÉTUDES D'INGÉNIEUR DE TUNIS
Kamel.Brahim@ipeit.rnu.tn

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ABSTRACT. In this paper, we provide a q-analogue of an open problem posed by Q. A. Ngô et al. in the paper, *Note on an integral inequality*, J. Inequal. Pure and Appl. Math., 7(4)(2006), Art. 120, by using analytic and elementary methods in Quantum Calculus.

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1. INTRODUCTION

In [9], Q.A. Ngô et al. studied an interesting integral inequality and proved the following result:

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 tdt, \quad \forall x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

Then, they proposed the following open problem: *Under what condition does the inequality*

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\beta f^\alpha(x)dx \tag{1.4}$$

hold for α and β?

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In view of the interest in this type of inequalities, much attention has been paid to the problem and many authors have extended the inequality to more general cases (see [1, 3, 7, 8]). In this paper, we shall discuss a q-analogue of Ngô's problem.

This paper is organized as follows: In Section 2, we present definitions and facts from the q-calculus necessary for understanding this paper. In Section 3, we discuss a q-analogue of the inequalities given in [9] and [3].

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [4] and [6]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \text{ if } x \neq 0, \tag{2.1}$$

$(D_q f)(0) = f'(0)$, provided $f'(0)$ exists.

The q-Jackson integral from 0 to a is defined by (see [5])

$$\int_0^a f(x) d_q x = (1-q)a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.2}$$

provided the sum converges absolutely.

The q-Jackson integral in a generic interval $[a,b]$ is given by (see [5])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.3}$$

We recall that for any function f, we have (see [6])

$$D_q \left(\int_a^x f(t) d_q t \right) = f(x). \tag{2.4}$$

If F is any anti q-derivative of the function f, namely $D_q F = f$, continuous at $x = 0$, then

$$\int_0^a f(x) d_q x = F(a) - F(0). \tag{2.5}$$

A q-analogue of the integration by parts formula is given by

$$\int_a^b f(x)(D_q g(x)) d_q x = f(a)g(a) - f(b)g(b) - \int_a^b (D_q f(x)) g(qx) d_q x. \tag{2.6}$$

Finally, we denote

$$[0,1]_q = \{q^k : \ k = 0, 1, 2, \ldots, \infty\}.$$

3. MAIN RESULTS

Let us begin with the following useful result:

Lemma 3.1 ([9] General Cauchy inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we always have*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}. \tag{3.1}$$

Theorem 3.2. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$\int_x^1 f^\beta(t)d_qt \geq \int_x^1 t^\beta d_qt, \quad \forall x \in [0,1]_q. \tag{3.2}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)d_qx \geq \int_0^1 x^\alpha f^\beta(x)d_qx, \tag{3.3}$$

holds for all positive real numbers $\alpha > 0$ and $\beta > 0$.

To prove Theorem 3.2, we need the following lemma.

Lemma 3.3. *Under the conditions of Theorem 3.2, we have*

$$\int_0^1 x^\alpha f^\beta(x)d_qx \geq \frac{1}{[\alpha+\beta+1]_q}. \tag{3.4}$$

Proof. By using a q-integration by parts, we obtain

$$\begin{aligned}\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)d_qt\right)d_qx &= \frac{1}{[\alpha]_q}\left[x^\alpha \int_x^1 f^\beta(t)d_qt\right]_{x=0}^{x=1} + \frac{q^\alpha}{[\alpha]_q}\int_0^1 x^\alpha f^\beta(x)d_qx \\ &= \frac{q^\alpha}{[\alpha]_q}\int_0^1 x^\alpha f^\beta(x)d_qx,\end{aligned}$$

which yields

$$\int_0^1 x^\alpha f^\beta(x)d_qx = \frac{[\alpha]_q}{q^\alpha}\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)d_qt\right)d_qx. \tag{3.5}$$

On the other hand, from condition (3.2), we get

$$\begin{aligned}\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)d_qt\right)d_qx &\geq \int_0^1 x^{\alpha-1}\left(\int_x^1 t^\beta d_qt\right)d_qx \\ &= \frac{1}{[\beta+1]_q}\int_0^1 (x^{\alpha-1} - x^{\alpha+\beta})d_qx \\ &= \frac{q^\alpha}{[\alpha]_q[\alpha+\beta+1]_q}.\end{aligned}$$

Therefore, from (3.5), we obtain

$$\int_0^1 x^\alpha f^\beta(x)d_qx \geq \frac{1}{[\alpha+\beta+1]_q}. \tag{3.6}$$

□

We now give the proof of Theorem 3.2.

Proof of Theorem 3.2. Using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta}x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{3.7}$$

which gives

$$\beta\int_0^1 f^{\alpha+\beta}(x)d_qx + \alpha\int_0^1 x^{\alpha+\beta}d_qx \geq (\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)d_qx. \tag{3.8}$$

Moreover, by using Lemma 3.3, we get

$$(3.9)\qquad \begin{aligned}(\alpha+\beta)\int_0^1 x^\alpha f^\beta(x)d_qx &= \alpha\int_0^1 x^\alpha f^\beta(x)d_qx + \beta\int_0^1 x^\alpha f^\beta(x)d_qx \\ &\geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta\int_0^1 x^\alpha f^\beta(x)d_qx.\end{aligned}$$

Then, from relation (3.8), we obtain

$$(3.10)\qquad \beta\int_0^1 f^{\alpha+\beta}(x)d_qx + \frac{\alpha}{[\alpha+\beta+1]_q} \geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta\int_0^1 x^\alpha f^\beta(x)d_qx,$$

which completes the proof. □

Taking $\beta = 1$ in Theorem 3.2, we obtain

Corollary 3.4. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$(3.11)\qquad \int_x^1 f(t)d_qt \geq \int_x^1 td_qt, \quad \forall x\in[0,1]_q.$$

Then the inequality

$$(3.12)\qquad \int_0^1 f^{\alpha+1}(x)d_qx \geq \int_0^1 x^\alpha f(x)d_qx$$

holds for every positive real number $\alpha > 0$.

Theorem 3.5. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$(3.13)\qquad \int_x^1 f(t)d_qt \geq \int_x^1 td_qt, \quad \forall x\in[0,1]_q.$$

Then the inequality

$$(3.14)\qquad \int_0^1 f^{\alpha+1}(x)d_qx \geq \int_0^1 xf^\alpha(x)d_qx$$

holds for every positive real number $\alpha > 0$.

Proof. We have

$$(3.15)\qquad \forall x\in[0,1]_q, \quad (f^\alpha(x)-x^\alpha)(f(x)-x)\geq 0,$$

so

$$(3.16)\qquad f^{\alpha+1}(x)+x^{\alpha+1} \geq x^\alpha f(x) + xf^\alpha(x).$$

By integrating with some simple calculations we deduce that

$$(3.17)\qquad \int_0^1 f^{\alpha+1}(x)d_qx + \frac{1}{[\alpha+2]_q} \geq \int_0^1 x^\alpha f(x)d_qx + \int_0^1 xf^\alpha(x)d_qx.$$

Then, from Lemma 3.3 for $\beta = 1$, the result follows. □

Theorem 3.6. *Let f be a nonnegative function defined on $[0,1]_q$ satisfying*

$$(3.18)\qquad \int_x^1 f(t)d_qt \geq \int_x^1 td_qt, \quad \forall x\in[0,1]_q.$$

Then the inequality

$$(3.19)\qquad \int_0^1 f^{\alpha+\beta}(x)d_qx \geq \int_0^1 x^\alpha f^\beta(x)d_qx$$

holds for all real numbers $\alpha > 0$ *and* $\beta \geq 1$.

Lemma 3.7. *Under the conditions of Theorem 3.6, we have*

$$\int_0^1 x^\alpha f^\beta(x) d_q x \geq \frac{1}{[\alpha+\beta+1]_q} \tag{3.20}$$

for all real numbers $\alpha > 0$ *and* $\beta \geq 1$.

Proof. Using Lemma 3.1, we obtain

$$\frac{1}{\beta} f^\beta(x) + \frac{\beta-1}{\beta} x^\beta \geq x^{\beta-1} f(x), \tag{3.21}$$

which implies

$$\int_0^1 x^\alpha f^\beta(x) d_q x + (\beta-1)\int_0^1 x^{\alpha+\beta} d_q x \geq \beta \int_0^1 x^{\alpha+\beta-1} f(x) d_q x. \tag{3.22}$$

Therefore, from Lemma 3.3, we get

$$\int_0^1 x^\alpha f^\beta(x) d_q x + \frac{\beta-1}{[\alpha+\beta+1]_q} \geq \frac{\beta}{[\alpha+\beta+1]_q}. \tag{3.23}$$

Thus (3.20) is proved. □

We now give the proof of Theorem 3.6.

Proof of Theorem 3.6. By using Lemma 3.1, we obtain

$$\frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{3.24}$$

which implies

$$\beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha+\beta+1]_q} \geq (\alpha+\beta) \int_0^1 x^\alpha f^\beta(x) d_q x. \tag{3.25}$$

Then, from Lemma 3.7, we obtain

$$\beta \int_0^1 f^{\alpha+\beta}(x) d_q x + \frac{\alpha}{[\alpha+\beta+1]_q} \geq \frac{\alpha}{[\alpha+\beta+1]_q} + \beta \int_0^1 x^\alpha f^\beta(x) d_q x, \tag{3.26}$$

which completes the proof. □

REFERENCES

[1] L. BOUGOFFA, Note on an open problem, *J. Inequal. Pure and Appl. Math.*, **8**(2) (2007), Art. 58. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=871`].

[2] L. BOUGOFFA, Corrigendum of the paper entitled: Note on an open problem, *J. Inequal. Pure and Appl. Math.*, **8**(4) (2007), Art. 121. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=910`].

[3] K. BOUKERRIOUA AND A. GUEZANE-LAKOUD, On an open question regarding an integral inequality, *J. Inequal. Pure and Appl. Math.*, **8**(3) (2007), Art. 77. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=885`].

[4] G. GASPER AND M. RAHMAN, *Basic Hypergeometric Series*, 2nd Edition, (2004), Encyclopedia of Mathematics and Its Applications, 96, Cambridge University Press, Cambridge.

[5] F.H. JACKSON, On q-definite integrals, *Quarterly Journal of Pure and Applied Mathematics*, **41** (1910), 193–203.

[6] V.G. KAC AND P. CHEUNG, *Quantum Calculus*, Universitext, Springer-Verlag, New York, (2002).

[7] W.J. LIU, C.C LI AND J.W. DONG, On an open problem concerning an integral inequality, *J. Inequal. Pure and Appl. Math.*, **8**(3) (2007), Art. 74. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=882`].

[8] W.J. LIU, G.S. CHENG AND C.C LI, Further development of an open problem, *J. Inequal. Pure and Appl. Math.*, **9**(1) (2008), Art. 14. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=952`].

[9] Q.A. NGÔ, D.D. THANG, T.T. DAT AND D.A. TUAN, Note on an integral inequality, *J. Inequal. Pure and Appl. Math.*, **7**(4) (2006), Art. 120. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=737`].