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ON THE SHARPENED HEISENBERG-WEYL INEQUALITY

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ABSTRACT. The well-known *second order moment Heisenberg-Weyl inequality (or uncertainty relation)* in Fourier Analysis states: Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a random real variable x such that $f \in L^2(\mathbb{R})$. Then the product of the second moment of the random real x for $|f|^2$ and the second moment of the random real ξ for $\left|\hat{f}\right|^2$ is at least $E_{|f|^2} \big/ 4\pi$, where $\hat{f}$ is the Fourier transform of f, such that $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\,dx$, $f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\,d\xi$, and $E_{|f|^2} = \int_{\mathbb{R}} |f(x)|^2\,dx$.

This uncertainty relation is well-known in classical quantum mechanics. In 2004, the author generalized the afore-mentioned result to *higher order moments* and in 2005, he investigated a Heisenberg-Weyl *type* inequality *without Fourier transforms.* In this paper, a sharpened form of this generalized Heisenberg-Weyl inequality is established *in Fourier analysis*. Afterwards, an open problem is proposed on some pertinent extremum principle.These results are useful in investigation of quantum mechanics.

Key words and phrases: Sharpened, Heisenberg-Weyl inequality, Gram determinant.

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1. INTRODUCTION

The serious question of certainty in science was high-lighted by Heisenberg, in 1927, via his *uncertainty principle* [1]. He demonstrated the impossibility of specifying simultaneously the position and the speed (or the momentum) of an electron within an atom. In 1933, according to Wiener [7] *a pair of transforms cannot both be very small.* This uncertainty principle was stated in 1925 by Wiener, according to Wiener's autobiography [8, p. 105-107], at a lecture in Göttingen. The following result of the *Heisenberg-Weyl Inequality* is credited to Pauli according to Weyl [6, p. 77, p. 393-394]. In 1928, according to Pauli [6] *the less the uncertainty in* $|f|^2$,

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the greater the uncertainty in $\left|\hat{f}\right|^2$, *and conversely*. This result does not actually appear in Heisenberg's seminal paper [1] (in 1927). The following second order moment Heisenberg-Weyl inequality provides a precise quantitative formulation of the above-mentioned uncertainty principle according to W. Pauli.

1.1. **Second Order Moment Heisenberg-Weyl Inequality ([3, 4, 5]):** *For any* $f \in L^2(\mathbb{R}),\ f : \mathbb{R} \to \mathbb{C}$, *such that*

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{|f|^2},$$

any fixed but arbitrary constants x_m, $\xi_m \in \mathbb{R}$, *and for the second order moments*

$$(\mu_2)_{|f|^2} = \sigma^2_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^2 |f(x)|^2 \, dx,$$

$$(\mu_2)_{\left|\hat{f}\right|^2} = \sigma^2_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^2 \left|\hat{f}(\xi)\right|^2 d\xi,$$

the second order moment Heisenberg-Weyl inequality

$$\sigma^2_{|f|^2} \cdot \sigma^2_{\left|\hat{f}\right|^2} \geq \frac{\|f\|_2^4}{16\pi^2}, \tag{H_1}$$

holds. Equality holds in (H_1) *if and only if the generalized Gaussians*

$$f(x) = c_0 \exp(2\pi i x \xi_m) \exp\left(-c(x - x_m)^2\right)$$

hold for some constants $c_0 \in \mathbb{C}$ *and* $c > 0$.

1.2. **Fourth Order Moment Heisenberg-Weyl Inequality ([3, pp. 26-27]):** *For any* $f \in L^2(\mathbb{R}),\ f : \mathbb{R} \to \mathbb{C}$, *such that* $\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = E_{|f|^2}$, *any fixed but arbitrary constants* x_m, $\xi_m \in \mathbb{R}$, *and for the fourth order moments*

$$(\mu_4)_{|f|^2} = \int_{\mathbb{R}} (x - x_m)^4 |f(x)|^2 \, dx \qquad \text{and}$$

$$(\mu_4)_{\left|\hat{f}\right|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^4 \left|\hat{f}(\xi)\right|^2 d\xi,$$

the fourth order moment Heisenberg-Weyl inequality

$$(\mu_4)_{|f|^2} \cdot (\mu_4)_{\left|\hat{f}\right|^2} \geq \frac{1}{64\pi^4} E^2_{2,f}, \tag{H_2}$$

holds, where

$$E_{2,f} = 2\int_{\mathbb{R}} \left[(1 - 4\pi^2 \xi_m^2 x_\delta^2) |f(x)|^2 - x_\delta^2 |f'(x)|^2 - 4\pi \xi_m x_\delta^2 Im(f(x)\overline{f'(x)})\right] dx,$$

with $x_\delta = x - x_m$, $\xi_\delta = \xi - \xi_m$, $\mathrm{Im}(\cdot)$ *is the imaginary part of* $(\cdot)$, *and* $|E_{2,f}| < \infty$.

The "inequality" (H_2) holds, unless $f(x) = 0$.

We note that if the ordinary differential equation of second order

$$f''_\alpha(x) = -2c_2 x_\delta^2 f_\alpha(x) \tag{ODE}$$

holds, with $\alpha = -2\pi\xi_m i$, $f_\alpha(x) = e^{\alpha x} f(x)$, and a constant $c_2 = \frac{1}{2}k_2^2 > 0$, $k_2 \in \mathbb{R}$ and $k_2 \neq 0$,then "equality" in (H_2) seems to occur. However, the solution of this differential equation (ODE), given by the function

$$f(x) = \sqrt{|x_\delta|} e^{2\pi i x \xi_m} \left[c_{20} J_{-1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right) + c_{21} J_{1/4}\left(\frac{1}{2}|k_2| x_\delta^2\right)\right],$$

in terms of the Bessel functions $J_{\pm 1/4}$ of the first kind of orders $\pm 1/4$, leads to a contradiction, because this $f \notin L^2(\mathbb{R})$. Furthermore, a limiting argument is required for this problem. For the proof of this inequality see [3].

It is *open* to investigate cases, where the integrand on the right-hand side of integral of $E_{2,f}$ will be nonnegative. For instance, for $x_m = \xi_m = 0$, this integrand is: $= |f(x)|^2 - x^2 |f'(x)|^2 \ (\geq 0)$. In 2004, we ([3, 4]) generalized the Heisenberg-Weyl inequality and in 2005 we [5] investigated a Heisenberg-Weyl type inequality without Fourier transforms. In this paper, a sharpened form of this generalized *Heisenberg-Weyl inequality* is established in Fourier analysis. *We state our* following *two pertinent propositions*. For their proofs see [3].

Proposition 1.1 (Generalized differential identity, [3]). *If $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, $0 \leq \left[\frac{k}{2}\right]$ is the greatest integer $\leq \frac{k}{2}$, $f^{(j)} = \frac{d^j}{dx^j} f$, and $\overline{(\cdot)}$ is the conjugate of $(\cdot)$, then*

$$(*) \qquad f(x)\,\overline{f^{(k)}}(x) + f^{(k)}(x)\,\bar{f}(x) = \sum_{i=0}^{\left[\frac{k}{2}\right]} (-1)^i \frac{k}{k-i} \binom{k-i}{i} \frac{d^{k-2i}}{dx^{k-2i}} \left| f^{(i)}(x) \right|^2,$$

holds for any fixed but arbitrary $k \in \mathbb{N} = \{1, 2, \ldots\}$, such that $0 \leq i \leq \left[\frac{k}{2}\right]$ for $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Proposition 1.2 (Lagrange type differential identity, [3]). *If $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, and $f_a = e^{ax} f$, where $a = -\beta i$, with $i = \sqrt{-1}$ and $\beta = 2\pi\xi_m$ for any fixed but arbitrary real constant ξ_m, as well as if*

$$A_{pk} = \binom{p}{k}^2 \beta^{2(p-k)}, \quad 0 \leq k \leq p,$$

and

$$B_{pkj} = s_{pk} \binom{p}{k} \binom{p}{j} \beta^{2p-j-k}, \quad 0 \leq k < j \leq p,$$

where $s_{pk} = (-1)^{p-k}$ $(0 \leq k \leq p)$, then

$$(LD) \qquad \left| f_a^{(p)} \right|^2 = \sum_{k=0}^{p} A_{pk} \left| f^{(k)} \right|^2 + 2 \sum_{0 \leq k \langle j \leq p} B_{pkj} Re \left(r_{pkj} f^{(k)} \overline{f^{(j)}} \right),$$

holds for any fixed but arbitrary $p \in \mathbb{N}_0$, where $\overline{(\cdot)}$ is the conjugate of $(\cdot)$, and $r_{pkj} = (-1)^{p - \frac{k+j}{2}}$ $(0 \leq k < j \leq p)$, and $\operatorname{Re}(\cdot)$ is the real part of $(\cdot)$.

2. SHARPENED HEISENBERG-WEYL INEQUALITY

We assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x (*or absolutely continuous in* $[-a, a]$, $a > 0$), and $w : \mathbb{R} \to \mathbb{R}$ a real valued weight function of x, as well as x_m, ξ_m any fixed but arbitrary real constants. Denote $f_a = e^{ax} f$, where $a = -2\pi\xi_m i$ with $i = \sqrt{-1}$, and $\hat{f}$ the Fourier transform of f, such that

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx \quad \text{and} \quad f(x) = \int_{\mathbb{R}} e^{2i\pi\xi x} \hat{f}(\xi)\, d\xi.$$

Also we denote

$$(\mu_{2p})_{w,|f|^2} = \int_{\mathbb{R}} w^2(x) (x - x_m)^{2p} |f(x)|^2\, dx,$$

$$(\mu_{2p})_{|\hat{f}|^2} = \int_{\mathbb{R}} (\xi - \xi_m)^{2p} \left| \hat{f}(\xi) \right|^2 d\xi$$

the $2p^{\text{th}}$ weighted moment of x for $|f|^2$ with weight function $w : \mathbb{R} \to \mathbb{R}$ and the $2p^{\text{th}}$ moment of ξ for $\left|\hat{f}\right|^2$, respectively. In addition, we denote

$$C_q = (-1)^q \frac{p}{p-q}\binom{p-q}{q}, \quad \text{if} \quad 0 \le q \le \left[\frac{p}{2}\right] \quad \left(= \text{the greatest integer} \le \frac{p}{2}\right),$$
$$I_{ql} = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \left|f^{(l)}(x)\right|^2 dx, \quad \text{if} \quad 0 \le l \le q \le \left[\frac{p}{2}\right],$$
$$I_{qkj} = (-1)^{p-2q} \int_{\mathbb{R}} w_p^{(p-2q)}(x) \operatorname{Re}\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x)\right) dx, \quad \text{if} \quad 0 \le k < j \le q \le \left[\frac{p}{2}\right],$$

where $r_{qkj} = (-1)^{q-\frac{k+j}{2}} \in \{\pm 1, \pm i\}$ and $w_p = (x - x_m)^p w$. We assume that all these integrals exist. Finally we denote

$$D_q = \sum_{l=0}^{q} A_{ql} I_{ql} + 2 \sum_{0 \le k \langle j \le q} B_{qkj} I_{qkj},$$

if $|D_q| < \infty$ holds for $0 \le q \le \left[\frac{p}{2}\right]$, where

$$A_{ql} = \binom{q}{l}^2 \beta^{2(q-l)}, \qquad B_{qkj} = s_{qk} \binom{q}{k}\binom{q}{j} \beta^{2q-j-k},$$

with $\beta = 2\pi\xi_m$, and $s_{qk} = (-1)^{q-k}$, and $E_{p,f} = \sum_{q=0}^{[p/2]} C_q D_q$, if $|E_{p,f}| < \infty$ holds for $p \in \mathbb{N}$.

In addition, we assume *the two conditions*:

$$\sum_{r=0}^{p-2q-1} (-1)^r \lim_{|x|\to\infty} w_p^{(r)}(x) \left(\left|f^{(l)}(x)\right|^2\right)^{(p-2q-r-1)} = 0, \tag{2.1}$$

for $0 \le l \le q \le \left[\frac{p}{2}\right]$, and

$$\sum_{r=0}^{p-2q-1} (-1)^r \lim_{|x|\to\infty} w_p^{(r)}(x) \left(Re\left(r_{qkj} f^{(k)}(x) \overline{f^{(j)}}(x)\right)\right)^{(p-2q-r-1)} = 0, \tag{2.2}$$

for $0 \le k < j \le q \le \left[\frac{p}{2}\right]$. Also,

$$|E_{p,f}^*| = \sqrt{E_{p,f}^2 + 4A^2} (\ge |E_{p,f}|),$$

where $A = \|u\| x_0 - \|v\| y_0$, with L^2−norm $\|\cdot\|^2 = \int_{\mathbb{R}} |\cdot|^2$, inner product $(|u|, |v|) = \int_{\mathbb{R}} |u| |v|$, and

$$u = w(x) x_\delta^p f_\alpha(x), \qquad v = f_\alpha^{(p)}(x);$$
$$x_0 = \int_{\mathbb{R}} |\nu(x)| |h(x)| dx, \qquad y_0 = \int_{\mathbb{R}} |u(x)| |h(x)| dx,$$

as well as

$$h(x) = \frac{1}{\sqrt[4]{2\pi}\sqrt{\sigma}} e^{-\frac{1}{4}\left(\frac{x-\mu}{\sigma}\right)^2},$$

where μ is the mean and σ the standard deviation, or

$$h(x) = \frac{1}{\sqrt[4]{n\pi}} \sqrt{\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}} \cdot \frac{1}{\left(1 + \frac{x^2}{n}\right)^{\frac{n+1}{4}}},$$

where $n \in \mathbb{N}$, and

$$\|h(x)\|^2 = \int_{\mathbb{R}} |h(x)|^2 dx = 1.$$

Theorem 2.1. *If $f \in L^2(\mathbb{R})$ (or absolutely continuous in $[-a,a]$, $a>0$), then*

$$\sqrt[2p]{(\mu_{2p})_{w,|f|^2}}\,\sqrt[2p]{(\mu_{2p})_{|\hat{f}|^2}} \geq \frac{1}{2\pi\sqrt[p]{2}}\sqrt[p]{\left|E^*_{p,f}\right|}, \tag{H_p^*}$$

holds for any fixed but arbitrary $p \in \mathbb{N}$.

Equality holds in (H_p^*) iff $v(x) = -2c_p u(x)$ holds for constants $c_p > 0$, and any fixed but arbitrary $p \in \mathbb{N}$; $c_p = k_p^2/2 > 0$, $k_p \in \mathbb{R}$ and $k_p \neq 0$, $p \in \mathbb{N}$, and $A = 0$, or

$$h(x) = c_{1p}u(x) + c_{2p}v(x)$$

and $x_0 = 0$, or *$y_0 = 0$,where c_{ip} $(i=1,2)$ are constants and $A^2 > 0$.*

Proof. In fact, from the generalized Plancherel-Parseval-Rayleigh identity [3, (GPP)], and the fact that $|e^{ax}| = 1$ as $a = -2\pi\xi_m i$, one gets

$$\begin{aligned}
M_p^* &= M_p - \frac{1}{(2\pi)^{2p}}A^2 \qquad (2.3)\\
&= (\mu_{2p})_{w,|f|^2}\cdot(\mu_{2p})_{|\hat{f}|^2} - \frac{1}{(2\pi)^{2p}}A^2\\
&= \left(\int_{\mathbb{R}} w^2(x)(x-x_m)^{2p}|f(x)|^2 dx\right)\cdot\left(\int_{\mathbb{R}}(\xi-\xi_m)^{2p}\left|\hat{f}(\xi)\right|^2 d\xi\right) - \frac{1}{(2\pi)^{2p}}A^2\\
&= \frac{1}{(2\pi)^{2p}}\left[\left(\int_{\mathbb{R}} w^2(x)(x-x_m)^{2p}|f_a(x)|^2 dx\right)\cdot\left(\int_{\mathbb{R}}\left|f_a^{(p)}(x)\right|^2 dx\right) - A^2\right]\\
&= \frac{1}{(2\pi)^{2p}}\left[\|u\|^2\|v\|^2 - A^2\right] \qquad (2.4)
\end{aligned}$$

with $u = w(x)x_\delta^p f_\alpha(x),\ v = f_\alpha^{(p)}(x)$.

From (2.3) – (2.4), *the Cauchy-Schwarz inequality* $(|u|,|v|) \leq \|u\|\,\|v\|$ and the non-negativeness of the following *Gram determinant* [2]

$$\begin{aligned}
0 &\leq \begin{vmatrix} \|u\|^2 & (|u|,|v|) & y_0 \\ (|v|,|u|) & \|v\|^2 & x_0 \\ y_0 & x_0 & 1 \end{vmatrix} \qquad (2.5)\\
&= \|u\|^2\|v\|^2 - (|u|,|v|)^2 - \left[\|u\|^2 x_0^2 - 2(|u|,|v|)x_0y_0 + \|v\|^2 y_0^2\right],\\
0 &\leq \|u\|^2\|v\|^2 - (|u|,|v|)^2 - A^2
\end{aligned}$$

with

$$A = \|u\|x_0 - \|v\|y_0, x_0 = \int_{\mathbb{R}}|\nu(x)|\,|h(x)|\,dx, y_0 = \int_{\mathbb{R}}|u(x)|\,|h(x)|\,dx,$$

and

$$\|h(x)\|^2 = \int_{\mathbb{R}}|h(x)|^2 dx = 1,$$

we find

$$M_p^* \geq \frac{1}{(2\pi)^{2p}}(|u|,|v|)^2 = \frac{1}{(2\pi)^{2p}}\left(\int_{\mathbb{R}} |u|\,|v|\right)^2 = \frac{1}{(2\pi)^{2p}}\left(\int_{\mathbb{R}} \left|w_p(x)\, f_a(x)\, f_a^{(p)}(x)\right| dx\right)^2, \tag{2.6}$$

where $w_p = (x-x_m)^p w$, and $f_a = e^{ax}f$. In general, if $\|h\| \neq 0$, then one gets

$$(u,v)^2 \leq \|u\|^2\,\|v\|^2 - R^2,$$

where $R = A/\|h\| = \|u\|\,x - \|v\|\,y$, such that $x = x_0/\|h\|,\ y = y_0/\|h\|$.

In this case, A *has to be replaced by* R in all the pertinent relations of this paper.

From (2.6) and *the complex inequality,* $|ab| \geq \frac{1}{2}\left(a\overline{b}+\overline{a}b\right)$ with $a = w_p(x)\, f_a(x),\ b = f_a^{(p)}(x)$, we get

$$M_p^* = \frac{1}{(2\pi)^{2p}}\left[\frac{1}{2}\int_{\mathbb{R}} w_p(x)\left(f_\alpha(x)\overline{f_\alpha^{(p)}(x)} + f_\alpha^{(p)}(x)\overline{f_\alpha(x)}\right)dx\right]^2. \tag{2.7}$$

From (2.7) and the generalized differential identity (*), one finds

$$M_p^* \geq \frac{1}{2^{2(p+1)}\pi^{2p}}\left[\int_{\mathbb{R}} w_p(x)\left(\sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}}\left|f_a^{(q)}(x)\right|^2\right)dx\right]^2. \tag{2.8}$$

From (2.8) and the Lagrange type differential identity (LD), we find

$$M_p^* \geq \frac{1}{2^{2(p+1)}\pi^{2p}}\left[\int_{\mathbb{R}} w_p(x)\left[\sum_{q=0}^{[p/2]} C_q \frac{d^{p-2q}}{dx^{p-2q}}\left(\sum_{l=0}^{q} A_{ql}\left|f^{(l)}(x)\right|^2 + 2\sum_{0\leq k\langle j\leq q} B_{qkj} Re\left(r_{qkj} f^{(k)}(x)\, \overline{f^{(j)}}(x)\right)\right)\right]dx\right]^2.$$

From the generalized integral identity [3], the two conditions (2.1) – (2.2), and that all the integrals exist, one gets

$$\int_{\mathbb{R}} w_p(x)\,\frac{d^{p-2q}}{dx^{p-2q}}\left|f^{(l)}(x)\right|^2 dx = (-1)^{p-2q}\int_{\mathbb{R}} w_p^{(p-2q)}(x)\left|f^{(l)}(x)\right|^2 dx = I_{ql},$$

as well as

$$\int_{\mathbb{R}} w_p(x)\,\frac{d^{p-2q}}{dx^{p-2q}}\,\mathrm{Re}\left(r_{qkj} f^{(k)}(x)\, \overline{f^{(j)}}(x)\right) = (-1)^{p-2q}\int_{\mathbb{R}} w_p^{(p-2q)}(x) Re\left(r_{qkj} f^{(k)}(x)\, \overline{f^{(j)}}(x)\right) = I_{qkj}.$$

Thus we find

$$M_p^* \geq \frac{1}{2^{2(p+1)}\pi^{2p}} \left[\sum_{q=0}^{[p/2]} C_q \left(\sum_{l=0}^{q} A_{ql} I_{ql} + 2 \sum_{0 \leq k \langle j \leq q} B_{qkj} I_{qkj} \right) \right]^2$$
$$= \frac{1}{2^{2(p+1)}\pi^{2p}} E_{p,f}^2,$$

where $E_{p,f} = \sum_{q=0}^{[p/2]} C_q D_q$, if $|E_{p,f}| < \infty$ holds, or *the sharpened moment uncertainty formula*

$$\sqrt[2p]{M_p} \geq \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{|E_{p,f}^*|} \qquad \left(\geq \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{|E_{p,f}|} \right),$$

where $M_p = M_p^* + \frac{1}{(2\pi)^{2p}} A^2$.

We note that the corresponding Gram matrix to the above Gram determinant is positive definite if and only if the above Gram determinant is positive if and only if u, v, h are linearly independent. Besides, the equality in (2.5) holds if and only if h is a linear combination of linearly independent u and v and $u = 0$ or $v = 0$, completing the proof of the above theorem. □

Let

$$(m_{2p})_{|f|^2} = \int_{\mathbb{R}} x^{2p} |f(x)|^2 \, dx$$

be the $2p^{\text{th}}$ moment of x for $|f|^2$ about the origin $x_m = 0$, and

$$(m_{2p})_{|\hat{f}|^2} = \int_{\mathbb{R}} \xi^{2p} \left| \hat{f}(\xi) \right|^2 d\xi$$

the $2p^{\text{th}}$ moment of ξ for $\left|\hat{f}\right|^2$ about the origin $\xi_m = 0$. Denote

$$\varepsilon_{p,q} = (-1)^{p-q} \frac{p}{p-q} \cdot \frac{p!}{(2q)!} \binom{p-q}{q},$$

if $p \in \mathbb{N}$ and $0 \leq q \leq \left[\frac{p}{2}\right]$.

Corollary 2.2. *Assume that $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function of a real variable x, $w = 1$, $x_m = \xi_m = 0$, and $\hat{f}$ is the Fourier transform of f, described in our theorem. If $f : \mathbb{R} \to \mathbb{C}$ (or absolutely continuous in $[-a, a]$, $a > 0$), then the following inequality*

$$(S_p) \qquad \sqrt[2p]{(m_{2p})_{|f|^2}} \sqrt[2p]{(m_{2p})_{|\hat{f}|^2}} \geq \frac{1}{2\pi \sqrt[p]{2}} \sqrt[p]{\left| \sum_{q=0}^{[p/2]} \varepsilon_{p,q} (m_{2q})_{|f^{(q)}|^2} \right|^2 + 4A^2},$$

holds for any fixed but arbitrary $p \in \mathbb{N}$ and $0 \leq q \leq \left[\frac{p}{2}\right]$, where

$$(m_{2q})_{|f^{(q)}|^2} = \int_{\mathbb{R}} x^{2q} \left| f^{(q)}(x) \right|^2 dx$$

and A is analogous to the one in the above theorem.

We consider the *extremum principle* (via (9.33) on p. 51 of [3]):

$$\text{(R)} \qquad R(p) \geq \frac{1}{2\pi}, \qquad p \in \mathbb{N}$$

for the corresponding "inequality" (H_p) [3, p. 22], $p \in \mathbb{N}$.

Problem 2.1. Employing our Theorem 8.1 on p. 20 of [3], the Gaussian function, the Euler gamma function Γ, and other related *special functions*, we established and explicitly proved *the above extremum principle* (R), where

$$R(p) = \frac{\Gamma\left(p+\frac{1}{2}\right)}{\left|\sum_{q=0}^{[p/2]}(-1)^{p-q}\frac{p}{p-q}\cdot\frac{p!}{(2q)!}\binom{p-q}{q}\Gamma_q\right|},$$

with

$$\Gamma_q = \sum_{k=0}^{[q/2]} 2^{2k}\binom{q}{2k}^2\Gamma^2\left(k+\frac{1}{2}\right)\Gamma\left(2q-2k+\frac{1}{2}\right) + 2\sum_{0\le k\le j\le[q/2]}(-1)^{k+j}2^{k+j}\binom{q}{2k}\binom{q}{2j} \times\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)\Gamma\left(2q-k-j+\frac{1}{2}\right),$$

$0 \le \left[\frac{q}{2}\right]$ is the greatest integer $\le \frac{q}{2}$ for $q \in \mathbb{N}\cup\{0\} = \mathbb{N}_0$, $\binom{p}{q} = \frac{p!}{q!(p-q)!}$ for $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ and $0 \le q \le p$, $p! = 1\cdot 2\cdot 3\cdot\cdots\cdot(p-1)\cdot p$ and $0! = 1$, as well as

$$\Gamma\left(p+\frac{1}{2}\right) = \frac{1}{2^{2p}}\cdot\frac{(2p)!}{p!}\sqrt{\pi},\ p\in\mathbb{N} \quad\text{and}\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Furthermore, by employing computer techniques, this principle was verified for $p = 1, 2, 3, \ldots, 32, 33$, as well. *It now remains open to give a second explicit proof of verification for the extremum principle (R) using only special functions techniques and without applying our Heisenberg-Pauli-Weyl inequality* [3].

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