Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 5, Article 172, 2006

A UNIFIED TREATMENT OF SOME SHARP INEQUALITIES

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Received 24 July, 2006; accepted 11 December, 2006
Communicated by S.S. Dragomir

ABSTRACT. A generalization of some recent sharp inequalities by N. Ujević is established. Applications in numerical integration are also considered.

Key words and phrases: Quadrature formula, Sharp error bounds, Generalization, Numerical integration.

2000 *Mathematics Subject Classification.* Primary 26D15.

1. INTRODUCTION

In [1] we can find a generalization of the pre-Grüss inequality as:

Lemma 1.1. *Let $f, g, \Psi \in L_2(a,b)$. Then we have*

$$S_\Psi(f,g)^2 \leq S_\Psi(f,f)S_\Psi(g,g), \tag{1.1}$$

where

$$S_\Psi(f,g) = \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \int_a^b g(t)\,dt - \frac{1}{\|\Psi\|_2^2}\int_a^b f(t)\Psi(t)\,dt \int_a^b g(t)\Psi(t)\,dt \tag{1.2}$$

and Ψ satisfies

$$\int_a^b \Psi(t)\,dt = 0, \tag{1.3}$$

while as usual, $\|\cdot\|_2$ is the norm in $L_2(a,b)$. i.e.,

$$\|\Psi\|_2^2 = \int_a^b \Psi^2(t)\,dt.$$

Using the above inequality, Ujević in [1] obtained the following interesting results:

ISSN (electronic): 1443-5756

195-06

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a,b)$. Then*

$$\left| f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)\,dt \right| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} C_1 \tag{1.4}$$

where

$$C_1 = \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - [Q(f;a,b)]^2 \right\}^{\frac{1}{2}} \tag{1.5}$$

and

$$Q(f;a,b) = \frac{2}{\sqrt{b-a}} \left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) - \frac{3}{b-a}\int_a^b f(t)\,dt \right]. \tag{1.6}$$

Theorem 1.3. *Let the assumptions of Theorem 1.2 hold. Then*

$$\left| \left(\frac{f(a)+f(b)}{2}\right)(b-a) - \int_a^b f(t)\,dt \right| \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} C_2, \tag{1.7}$$

where

$$C_2 = \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - [P(f;a,b)]^2 \right\}^{\frac{1}{2}} \tag{1.8}$$

and

$$P(f;a,b) = \frac{1}{\sqrt{b-a}} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) - \frac{6}{b-a}\int_a^b f(t)\,dt \right]. \tag{1.9}$$

Theorem 1.4. *Let the assumptions of Theorem 1.2 hold. Then*

$$\left| \frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4}(b-a) - \int_a^b f(t)\,dt \right| \le \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}} C_3, \tag{1.10}$$

where

$$\begin{aligned} C_3 &= \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - \frac{1}{b-a}\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b) \right)^2 \right\}^{\frac{1}{2}} \\ &= \left\{ \|f'\|_2^2 - \frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right) - f(a) \right]^2 - \frac{2}{b-a}\left[f(b) - f\left(\frac{a+b}{2}\right) \right]^2 \right\}^{\frac{1}{2}}. \end{aligned} \tag{1.11}$$

In [2], Ujević further proved that the above all inequalities are sharp.

In this paper, we will derive a new sharp inequality with a parameter for absolutely continuous functions with derivatives belonging to $L_2(a,b)$, which not only provides a unified treatment of all the above sharp inequalities, but also gives some other interesting results as special cases. Applications in numerical integration are also considered.

2. Main Results

Theorem 2.1. *Let the assumptions of Theorem 1.2 hold. Then for any $\theta \in [0,1]$ we have*

$$\begin{aligned} &\left| (b-a)\left[(1-\theta) f\left(\frac{a+b}{2}\right) + \theta \frac{f(a)+f(b)}{2} \right] - \int_a^b f(t)\,dt \right| \\ &\qquad\qquad \le \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} \sqrt{1-3\theta+3\theta^2}\, C(\theta), \end{aligned} \tag{2.1}$$

where

$$C(\theta)=\left\{\|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a}-[N(f;a,b;\theta)]^2\right\}^{\frac{1}{2}} \tag{2.2}$$

and

$$N(f;a,b;\theta)=\frac{2}{\sqrt{(1-3\theta+3\theta^2)(b-a)}} \times\left|(1-3\theta)f\left(\frac{a+b}{2}\right)+(2-3\theta)\frac{f(a)+f(b)}{2}-\frac{3-6\theta}{b-a}\int_a^b f(t)\,dt\right|. \tag{2.3}$$

The inequality (2.1) with (2.2) and (2.3) is sharp in the sense that the constant $\frac{1}{2\sqrt{3}}$ cannot be replaced by a smaller one.

Proof. Let us define the functions

$$p(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right],\\ t-b, & t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

and

$$\Psi(t)=\begin{cases} t-\left(a+\theta\frac{b-a}{2}\right), & t\in\left[a,\frac{a+b}{2}\right],\\ t-\left(b-\theta\frac{b-a}{2}\right), & t\in\left(\frac{a+b}{2},b\right],\end{cases}$$

where $\theta\in[0,1]$.

It is not difficult to verify that

$$\int_a^b p(t)\,dt=\int_a^b \Psi(t)\,dt=0. \tag{2.4}$$

i.e., Ψ satisfies the condition (1.3).

We also have

$$\|p\|_2^2=\int_a^b p^2(t)\,dt=\frac{(b-a)^3}{12} \tag{2.5}$$

and

$$\|\Psi\|_2^2=\int_a^b \Psi^2(t)\,dt=\frac{(b-a)^3}{12}(1-3\theta+3\theta^2). \tag{2.6}$$

We now calculate

$$\begin{aligned}\int_a^b p(t)\Psi(t)\,dt &=\int_a^{\frac{a+b}{2}}(t-a)\left(t-a-\theta\frac{b-a}{2}\right)dt+\int_{\frac{a+b}{2}}^b (t-b)\left(t-b+\theta\frac{b-a}{2}\right)dt\\ &=\left(\frac{1}{12}-\frac{\theta}{8}\right)(b-a)^3.\end{aligned} \tag{2.7}$$

Integrating by parts, we have

$$\int_a^b f'(t)p(t)\,dt=f\left(\frac{a+b}{2}\right)(b-a)-\int_a^b f(t)\,dt \tag{2.8}$$

and

$$\begin{aligned}
(2.9)\qquad &\int_a^b f'(t)\Psi(t)\,dt \\
&= \int_a^{\frac{a+b}{2}} \left(t-a-\theta\frac{b-a}{2}\right) f'(t)\,dt + \int_{\frac{a+b}{2}}^b \left(t-b+\theta\frac{b-a}{2}\right) f'(t)\,dt \\
&= (b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt.
\end{aligned}$$

From (2.4), (2.6) – (2.9) and (1.2) we get

$$\begin{aligned}
(2.10)\qquad &S_\Psi(f',p) \\
&= \int_a^b f'(t)p(t)\,dt - \frac{1}{b-a}\int_a^b f'(t)\,dt\int_a^b p(t)\,dt \\
&\qquad - \frac{1}{\|\Psi\|_2^2}\int_a^b f'(t)\Psi(t)\,dt\int_a^b p(t)\Psi(t)\,dt \\
&= f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t)\,dt - \frac{2-3\theta}{2(1-3\theta+3\theta^2)} \\
&\qquad \times\left\{(b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right] - \int_a^b f(t)\,dt\right\} \\
&= \frac{\theta}{2(1-3\theta+3\theta^2)}\left\{(b-a)\left[(1-3\theta)f\left(\frac{a+b}{2}\right)\right.\right. \\
&\qquad \left.\left. +\ (2-3\theta)\frac{f(a)+f(b)}{2}\right] - (3-6\theta)\int_a^b f(t)\,dt\right\}.
\end{aligned}$$

From (2.4) – (2.7) and (1.2) we also have

$$\begin{aligned}
(2.11)\qquad S_\Psi(p,p) &= \|p\|_2^2 - \frac{1}{b-a}\left(\int_a^b p(t)\,dt\right)^2 - \frac{1}{\|\Psi\|_2^2}\left(\int_a^b p(t)\Psi(t)\,dt\right)^2 \\
&= \frac{\theta^2(b-a)^3}{16(1-3\theta+3\theta^2)}
\end{aligned}$$

and

$$\begin{aligned}
(2.12)\qquad S_\Psi(f',f') &= \|f'\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'(t)\,dt\right)^2 - \frac{1}{\|\Psi\|_2^2}\left(\int_a^b f'(t)\Psi(t)\,dt\right)^2 \\
&= \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - \frac{12}{(1-3\theta+3\theta^2)(b-a)} \\
&\qquad \times\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right]^2
\end{aligned}$$

Thus from (2.10) – (2.12) and (1.1) we can easily get

$$\left|(b-a)\left[(1-3\theta)f\left(\frac{a+b}{2}\right)+(2-3\theta)\frac{f(a)+f(b)}{2}\right]-(3-6\theta)\int_a^b f(t)\,dt\right|^2 \\ \leq \frac{(1-3\theta+3\theta^2)(b-a)^3}{4}\left\{\|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a}-\frac{12}{(1-3\theta+3\theta^2)(b-a)} \\ \times\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]^2\right\}. \tag{2.13}$$

It is equivalent to

$$3(b-a)^2\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]^2 \\ \leq \frac{1-3\theta+3\theta^2}{4}(b-a)^3\left\{\|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a}-\frac{4}{(1-3\theta+3\theta^2)(b-a)} \\ \times\left|(1-3\theta)f\left(\frac{a+b}{2}\right)+(2-3\theta)\frac{f(a)+f(b)}{2}-\frac{3-6\theta}{b-a}\int_a^b f(t)\,dt\right|^2\right\}. \tag{2.14}$$

Consequently, inequality (2.1) with (2.2) and (2.3) follow from (2.14).

In order to prove that the inequality (2.1) with (2.2) and (2.3) is sharp for any $\theta \in [0,1]$, we define the function

$$f(t)=\begin{cases}\frac{1}{2}t^2-\frac{\theta}{2}t, & t\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}t^2-\left(1-\frac{\theta}{2}\right)t+\frac{1-\theta}{2}, & t\in\left(\frac{1}{2},1\right]\end{cases} \tag{2.15}$$

The function given in (2.15) is absolutely continuous since it is a continuous piecewise polynomial function.

We now suppose that (2.1) holds with a constant $C>0$ as

$$\left|(b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right)+\theta\frac{f(a)+f(b)}{2}\right]-\int_a^b f(t)\,dt\right| \\ \leq C(b-a)^{\frac{3}{2}}\sqrt{1-3\theta+3\theta^2}\,C(\theta), \tag{2.16}$$

where $C(\theta)$ is as defined in (2.2) and (2.3).

Choosing $a=0$, $b=1$, and f defined in (2.15), we get

$$\int_0^1 f(t)\,dt=\frac{1}{24}-\frac{\theta}{8},$$

$$f(0)=f(1)=0,\qquad f\left(\frac{1}{2}\right)=\frac{1}{8}-\frac{\theta}{4},$$

$$\int_0^1 (f'(t))^2\,dt=\frac{1-3\theta+3\theta^2}{12}$$

and

$$N(f;a,b;\theta)=0$$

such that the left-hand side becomes

$$L.H.S.(2.16)=\frac{1-3\theta+3\theta^2}{12}. \tag{2.17}$$

We also find that the right-hand side is

$$R.H.S.(2.16) = \frac{C(1 - 3\theta + 3\theta^2)}{2\sqrt{3}}. \tag{2.18}$$

From (2.16) – (2.18), we find that $C \geq \frac{1}{2\sqrt{3}}$, proving that the constant $\frac{1}{2\sqrt{3}}$ is the best possible in (2.1). □

Remark 2.2. If we take $\theta = 0$, $\theta = 1$ and $\theta = \frac{1}{2}$ in (2.1) with (2.2) and (2.3), we recapture the sharp midpoint type inequality (1.4) with (1.5) and (1.6), the sharp trapezoid type inequality (1.7) with (1.8) and (1.9) and the sharp averaged midpoint-trapezoid type inequality (1.10) with (1.11), respectively. Thus Theorem 2.1 may be regarded as a generalization of Theorem 1.2, Theorem 1.3 and Theorem 1.4.

Remark 2.3. If we take $\theta = \frac{1}{3}$, we get a sharp Simpson type inequality as

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int_a^b f(t)\,dt \right| \leq \frac{(b-a)^{\frac{3}{2}}}{6} C_4, \tag{2.19}$$

where

$$C_4 = \left\{ \|f'\|_2^2 - \frac{[f(b) - f(a)]^2}{b-a} - [R(f;a,b)]^2 \right\}^{\frac{1}{2}} \tag{2.20}$$

and

$$R(f;a,b) = N\left(f;a,b;\frac{1}{3}\right) = \frac{2\sqrt{3}}{\sqrt{b-a}} \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \right|. \tag{2.21}$$

3. Applications in Numerical Integration

We restrict further considerations to the averaged midpoint-trapezoid quadrature rule. We also emphasize that similar considerations may be made for all the quadrature rules considered in the previous section.

Theorem 3.1. *Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 1.4 hold. Then we have*

$$\left| \int_a^b f(t)\,dt - \frac{h}{4} \sum_{i=0}^{n-1} \left[f(x_i) + 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1}) \right] \right| \leq \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}n} \delta_n(f) \leq \frac{(b-a)^{\frac{3}{2}}}{4\sqrt{3}n} \lambda_n(f). \tag{3.1}$$

where

$$\delta_n(f) = \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - \frac{1}{b-a}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i) - 2\sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \right]^2 \right\}^{\frac{1}{2}} \tag{3.2}$$

and

$$\lambda_n(f) = \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} \right\}^{\frac{1}{2}}. \tag{3.3}$$

Proof. From (1.10) and (1.11) in Theorem 1.4 we obtain

$$\left|\frac{h}{4}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]-\int_{x_i}^{x_{i+1}}f(t)\,dt\right| \le \frac{h^{\frac{3}{2}}}{4\sqrt{3}}\left\{\int_{x_i}^{x_{i+i}}(f'(t))^2\,dt-\frac{1}{h}[f(x_{i+1})-f(x_i)]^2 - \frac{1}{h}\left[f(x_i)-2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]^2\right\}^{\frac{1}{2}}. \tag{3.4}$$

By summing (3.4) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\left|\int_a^b f(t)\,dt-\frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]\right| \le \frac{h^{\frac{3}{2}}}{4\sqrt{3}}\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+i}}(f'(t))^2\,dt-\frac{1}{h}[f(x_{i+1})-f(x_i)]^2 - \frac{1}{h}\left[f(x_i)-2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]^2\right\}^{\frac{1}{2}}. \tag{3.5}$$

By using the Cauchy inequality twice, we can easily obtain

$$\begin{aligned}\sum_{i=0}^{n-1}&\left\{\int_{x_i}^{x_{i+1}}(f'(t))^2\,dt-\frac{1}{h}[f(x_{i+1})-f(x_i)]^2 -\frac{1}{h}\left[f(x_i)-2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]^2\right\}^{\frac{1}{2}}\\ &\le \sqrt{n}\left\{\|f'\|_2^2-\frac{n}{b-a}\sum_{i=0}^{n-1}[f(x_{i+1})-f(x_i)]^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}\left[f(x_i)-2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]^2\right\}^{\frac{1}{2}}\\ &\le \sqrt{n}\left\{\|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a} - \frac{1}{b-a}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1}f(x_i)-2\sum_{i=0}^{n-1}f\left(\frac{x_i+x_{i+1}}{2}\right)\right]^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.6}$$

Consequently, the inequality (3.1) with (3.2) and (3.3) follow from (3.5) and (3.6). □

Remark 3.2. It should be noticed that Theorem 3.1 seems to be a revision and an improvement of the corresponding result in [2, Theorem 6.1].

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