A CERTAIN CLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS DEFINED BY MEANS OF A LINEAR OPERATOR

DING-GONG YANG
Department of Mathematics
Suzhou University
Suzhou, Jiangsu 215006,
P.R. China.

N-ENG XU
Department of Mathematics
Changshu Institute of Technology
Changshu, Jiangsu 215500,
P.R. China.
EMail: xuneng11@pub.sz.jsinfo.net

SHIGEYOSHI OWA
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
EMail: owa@math.kindai.ac.jp

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Abstract: Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), we introduce a class $Q_p(a, c; h)$ of analytic and multivalent functions in the open unit disk. An inclusion relation and a convolution property for the class $Q_p(a, c; h)$ are presented. Some integral-preserving properties are also given.

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1. Introduction and Preliminaries

Let the functions

$$f(z) = \sum_{k=0}^{\infty} a_k z^{p+k} \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^{p+k} \; (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

be analytic in the open unit disk $U = \{z : |z| < 1\}$. Then the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^{p+k} = (g * f)(z). \tag{1.1}$$

Let A_p denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_k z^{p+k} \quad (p \in \mathbb{N}), \tag{1.2}$$

which are analytic in U. A function $f(z) \in A_p$ is said to be in the class $S_p^*(\alpha)$ if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > p\alpha \quad (z \in U) \tag{1.3}$$

for some $\alpha(\alpha < 1)$. When $0 \le \alpha < 1$, $S_p^*(\alpha)$ is the class of p-valently starlike functions of order α in U. Also we write $A_1 = A$ and $S_1^*(\alpha) = S^*(\alpha)$. A function $f(z) \in A$ is said to be prestarlike of order $\alpha(\alpha < 1)$ in U if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha). \tag{1.4}$$

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We denote this class by $R(\alpha)$ (see [9]). It is clear that a function $f(z) \in A$ is in the class $R(0)$ if and only if $f(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

We now define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (z \in U), \tag{1.5}$$

where

$$c \notin \{0, -1, -2, \dots\} \quad \text{and} \quad (x)_k = x(x+1)\cdots(x+k-1) \quad (k \in \mathbb{N}).$$

Corresponding to the function $\varphi_p(a, c; z)$, Saitoh [10] introduced and studied a linear operator $L_p(a, c)$ on A_p by the following Hadamard product (or convolution):

$$L_p(a, c)f(z) = \varphi_p(a, c; z) * f(z) \quad (f(z) \in A_p). \tag{1.6}$$

For $p = 1$, $L_1(a, c)$ on A was first defined by Carlson and Shaffer [1]. We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ was introduced by Dziok and Srivastava [2].

It is known [10] that

$$z(L_p(a, c)f(z))' = aL_p(a+1, c)f(z) - (a-p)L_p(a, c)f(z) \quad (f(z) \in A_p). \tag{1.7}$$

Setting $a = n + p > 0$ and $c = 1$ in (1.6), we have

$$L_p(n+p, 1)f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) = D^{n+p-1}f(z) \quad (f(z) \in A_p). \tag{1.8}$$

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The operator D^{n+p-1} when $p = 1$ was first introduced by Ruscheweyh [8], and D^{n+p-1} was introduced by Goel and Sohi [3]. Thus we name D^{n+p-1} as the Ruscheweyh derivative of $(n+p-1)$th order.

For functions $f(z)$ and $g(z)$ analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in U).$$

Furthermore, if the function $g(z)$ is univalent in U, then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

Let P be the class of analytic functions $h(z)$ with $h(0) = p$, which are convex univalent in U and for which

$$\operatorname{Re} h(z) > 0 \quad (z \in U).$$

In this paper we introduce and investigate the following subclass of A_p.

Definition 1.1. *A function $f(z) \in A_p$ is said to be in the class $Q_p(a, c; h)$ if it satisfies*

$$\frac{L_p(a+1, c) f(z)}{L_p(a, c) f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}, \tag{1.9}$$

where

$$a \neq 0, \quad c \notin \{0, -1, -2, \dots\} \quad \textit{and} \quad h(z) \in P. \tag{1.10}$$

It is easy to see that, if $f(z) \in Q_p(a, c; h)$, then $L_p(a, c) f(z) \in S_p^*(0)$.

For $a = n + p \, (n > -p)$, $c = 1$ and

$$h(z) = p + \frac{(A-B)z}{1+Bz} \quad (-1 \leq B < A \leq 1), \tag{1.11}$$

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Yang [12] introduced and studied the class

$$Q_p(n+p,1;h) = S_{n,p}(A,B).$$

For $h(z)$ given by (1.11), the class

$$Q_p(a,c;h) = H_{a,c,p}(A,B) \tag{1.12}$$

has been considered by Liu and Owa [5].

For $p = 1$, $A = 1-2\alpha$ $(0 \le \alpha < 1)$ and $B = -1$, Kim and Srivastava [4] have shown some properties of the class $H_{a,c,1}(1-2\alpha,-1)$.

In the present paper, we shall establish an inclusion relation and a convolution property for the class $Q_p(a,c;h)$. Integral transforms of functions in this class are also discussed. We observe that the proof of each of the results in [5] is much akin to that of the corresponding assertion made by Yang [12] in the case of $a = n+p$ and $c = 1$. However, the methods used in [5, 12] do not work for the general function class $Q_p(a,c;h)$.

We need the following lemmas in order to derive our main results for the class $Q_p(a,c;h)$.

Lemma 1.2 (Ruscheweyh [9]). *Let $\alpha < 1$, $f(z) \in S^*(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in U,*

$$\frac{g*(fF)}{g*f}(U) \subset \overline{\mathrm{co}}(F(U)),$$

where $\overline{\mathrm{co}}(F(U))$ denotes the closed convex hull of $F(U)$.

Lemma 1.3 (Miller and Mocanu [6]). *Let β $(\beta \ne 0)$ and γ be complex numbers and let $h(z)$ be analytic and convex univalent in U with*

$$\mathrm{Re}(\beta h(z) + \gamma) > 0 \quad (z \in U).$$

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If $q(z)$ *is analytic in* U *with* $q(0) = h(0)$*, then the subordination*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z)$$

implies that $q(z) \prec h(z)$*.*

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2. Main Results

Theorem 2.1. *Let* $h(z) \in P$ *and*

$$\operatorname{Re} h(z) > \beta \quad (z \in U; 0 \leq \beta < p). \tag{2.1}$$

If

$$0 < a_1 < a_2 \quad \textit{and} \quad a_2 \geq 2(p - \beta), \tag{2.2}$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. Define

$$g(z) = z + \sum_{k=1}^{\infty} \frac{(a_1)_k}{(a_2)_k} z^{k+1} \quad (z \in U; 0 < a_1 < a_2).$$

Then

$$\frac{\varphi_p(a_1, a_2; z)}{z^{p-1}} = g(z) \in A, \tag{2.3}$$

where $\varphi_p(a_1, a_2; z)$ is defined as in (1.5), and

$$\frac{z}{(1-z)^{a_2}} * g(z) = \frac{z}{(1-z)^{a_1}}. \tag{2.4}$$

From (2.4) we have

$$\frac{z}{(1-z)^{a_2}} * g(z) \in S^*\left(1 - \frac{a_1}{2}\right) \subset S^*\left(1 - \frac{a_2}{2}\right)$$

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for $0 < a_1 < a_2$, which implies that

$$g(z) \in R\left(1 - \frac{a_2}{2}\right). \tag{2.5}$$

Since

$$L_p(a_1, c)f(z) = \varphi_p(a_1, a_2; z) * L_p(a_2, c)f(z) \quad (f(z) \in A_p), \tag{2.6}$$

we deduce from (1.7) and (2.6) that

$$\begin{aligned} a_1 L_p(a_1 + 1, c)f(z) & \\ &= z\left(L_p(a_1, c)f(z)\right)' + (a_1 - p)L_p(a_1, c)f(z) \\ &= \varphi_p(a_1, a_2; z) * \left(z(L_p(a_2, c)f(z))' + (a_1 - p)L_p(a_2, c)f(z)\right) \\ &= \varphi_p(a_1, a_2; z) * \left(a_2 L_p(a_2 + 1, c)f(z) + (a_1 - a_2)L_p(a_2, c)f(z)\right). \end{aligned} \tag{2.7}$$

By using (2.3), (2.6) and (2.7), we find that

$$\begin{aligned} \frac{L_p(a_1 + 1, c)f(z)}{L_p(a_1, c)f(z)} & \\ &= \frac{(z^{p-1}g(z)) * \left(\frac{a_2}{a_1} L_p(a_2 + 1, c)f(z) + \left(1 - \frac{a_2}{a_1}\right) L_p(a_2, c)f(z)\right)}{(z^{p-1}g(z)) * L_p(a_2, c)f(z)} \\ &= \frac{g(z) * \left(\frac{a_2}{a_1} \frac{L_p(a_2+1,c)f(z)}{z^{p-1}} + \left(1 - \frac{a_2}{a_1}\right) \frac{L_p(a_2,c)f(z)}{z^{p-1}}\right)}{g(z) * \frac{L_p(a_2,c)f(z)}{z^{p-1}}} \\ &= \frac{g(z) * (q(z)F(z))}{g(z) * q(z)} \quad (f(z) \in A_p), \end{aligned} \tag{2.8}$$

where

$$q(z) = \frac{L_p(a_2, c)f(z)}{z^{p-1}} \in A$$

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and

$$F(z) = \frac{a_2 L_p(a_2+1,c)f(z)}{a_1 L_p(a_2,c)f(z)} + 1 - \frac{a_2}{a_1}.$$

Let $f(z) \in Q_p(a_2, c; h)$. Then

$$\begin{aligned} F(z) &\prec \frac{a_2}{a_1}\left(1 - \frac{p}{a_2} + \frac{h(z)}{a_2}\right) + 1 - \frac{a_2}{a_1} \\ &= 1 - \frac{p}{a_1} + \frac{h(z)}{a_1} = h_1(z) \quad \text{(say)}, \end{aligned} \tag{2.9}$$

where $h_1(z)$ is convex univalent in U, and, by (1.7),

$$\begin{aligned} \frac{zq'(z)}{q(z)} &= \frac{z(L_p(a_2,c)f(z))'}{L_p(a_2,c)f(z)} + 1 - p \\ &= a_2 \frac{L_p(a_2+1,c)f(z)}{L_p(a_2,c)f(z)} + 1 - a_2 \\ &\prec 1 - p + h(z). \end{aligned} \tag{2.10}$$

By using (2.1), (2.2) and (2.10), we get

$$\operatorname{Re} \frac{zq'(z)}{q(z)} > 1 - p + \beta \geq 1 - \frac{a_2}{2} \quad (z \in U),$$

that is,

$$q(z) \in S^*\left(1 - \frac{a_2}{2}\right). \tag{2.11}$$

Consequently, in view of (2.5), (2.8), (2.9) and (2.11), an application of Lemma 1.2 yields

$$\frac{L_p(a_1+1,c)f(z)}{L_p(a_1,c)f(z)} \prec h_1(z).$$

Thus $f(z) \in Q_p(a_1, c; h)$ and the proof of Theorem 2.1 is completed. □

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By carefully selecting the function $h(z)$ involved in Theorem 2.1, we can obtain a number of useful consequences.

Corollary 2.2. *Let*

$$h(z) = p - 1 + \left(\frac{1+Az}{1+Bz}\right)^{\gamma} \quad (z \in U;\ 0 < \gamma \leq 1;\ -1 \leq B < A \leq 1). \tag{2.12}$$

If

$$0 < a_1 < a_2 \quad \textit{and} \quad a_2 \geq 2\left(1 - \left(\frac{1-A}{1-B}\right)^{\gamma}\right),$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The analytic function $h(z)$ defined by (2.12) is convex univalent in U (cf. [11]), $h(0) = p$, and $h(U)$ is symmetric with respect to the real axis. Thus $h(z) \in P$ and

$$\operatorname{Re} h(z) > \beta = h(-1) = p - 1 + \left(\frac{1-A}{1-B}\right)^{\gamma} \geq 0 \quad (z \in U).$$

Hence the desired result follows from Theorem 2.1 at once. □

If we let $\gamma = 1$, then Corollary 2.2 yields the following.

Corollary 2.3. *Let $h(z)$ be given by (1.11). If a, A and $B(-1 \leq B < A \leq 1)$ satisfy either*

(i) $a \geq 1 - 2\left(\frac{1-A}{1-B}\right) > 0$
or

(ii) $a > 0 \geq 1 - 2\left(\frac{1-A}{1-B}\right)$,
then

$$Q_p(a+1, c; h) \subset Q_p(a, c; h).$$

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Using Jack's Lemma, Liu and Owa [5, Theorem 1] proved that, if $a \geq \frac{A-B}{1-B}$, then

$$H_{a+1,c,p}(A,B) \subset H_{a,c,p}(A,B).$$

Since

$$\frac{A-B}{1-B} \geq 1 - 2\left(\frac{1-A}{1-B}\right) \quad (-1 \leq B < A \leq 1)$$

and the equality occurs only when $A = 1$, we see that Corollary 2.3 is better than the result of [5].

Corollary 2.4. *Let*

$$h(z) = p + \sum_{k=1}^{\infty} \left(\frac{\gamma+1}{\gamma+k}\right) \delta^k z^k \quad (z \in U; 0 < \delta \leq 1; \gamma \geq 0). \tag{2.13}$$

If

$$0 < a_1 < a_2 \quad \textit{and} \quad a_2 \geq 2 \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\gamma+1}{\gamma+k}\right) \delta^k,$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The function $h(z)$ defined by (2.13) is in the class P (cf. [8]) and satisfies $h(\overline{z}) = \overline{h(z)}$. Thus

$$\operatorname{Re} h(z) > \beta = h(-1) = p + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\gamma+1}{\gamma+k}\right) \delta^k > p - \delta \geq 0 \quad (z \in U).$$

Therefore we have the corollary by using Theorem 2.1. □

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Corollary 2.5. *Let*

$$h(z) = p + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{\gamma z}}{1-\sqrt{\gamma z}}\right)\right)^2 \quad (z \in U; 0 < \gamma \leq 1). \tag{2.14}$$

If

$$0 < a_1 < a_2 \quad \text{and} \quad a_2 \geq \frac{16}{\pi^2}\left(\arctan\sqrt{\gamma}\right)^2,$$

then

$$Q_p(a_2, c; h) \subset Q_p(a_1, c; h).$$

Proof. The function $h(z)$ defined by (2.14) belongs to the class P (cf. [7]) and satisfies $h(\overline{z}) = \overline{h(z)}$. Thus

$$\operatorname{Re} h(z) > \beta = h(-1) = p - \frac{8}{\pi^2}\left(\arctan\sqrt{\gamma}\right)^2 \geq p - \frac{1}{2} > 0 \quad (z \in U).$$

Hence an application of Theorem 2.1 yields the desired result. □

For $\gamma = 1$, Corollary 2.5 leads to

Corollary 2.6. *Let*

$$h(z) = p + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \quad (z \in U).$$

Then, for $a > 0$,

$$Q_p(a+1, c; h) \subset Q_p(a, c; h).$$

Theorem 2.7. *Let $h(z) \in P$ and*

$$\operatorname{Re} h(z) > p - 1 + \alpha \quad (z \in U; \alpha < 1). \tag{2.15}$$

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If $f(z) \in Q_p(a, c; h)$,

$$g(z) \in A_p \quad \textit{and} \quad \frac{g(z)}{z^{p-1}} \in R(\alpha) \quad (\alpha < 1), \tag{2.16}$$

then

$$(f * g)(z) \in Q_p(a, c; h).$$

Proof. Let $f(z) \in Q_p(a, c; h)$ and suppose that

$$q(z) = \frac{L_p(a, c)f(z)}{z^{p-1}}. \tag{2.17}$$

Then

$$F(z) = \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}, \tag{2.18}$$

$q(z) \in A$ and

$$\frac{zq'(z)}{q(z)} \prec 1 - p + h(z) \tag{2.19}$$

(see (2.10) used in the proof of Theorem 2.1). By (2.15) and (2.19), we see that

$$q(z) \in S^*(\alpha). \tag{2.20}$$

For $g(z) \in A_p$, it follows from (2.17) and (2.18) that

$$\begin{aligned} \frac{L_p(a+1, c)(f * g)(z)}{L_p(a, c)(f * g)(z)} &= \frac{g(z) * L_p(a+1, c)f(z)}{g(z) * L_p(a, c)f(z)} \\ &= \frac{\frac{g(z)}{z^{p-1}} * (q(z)F(z))}{\frac{g(z)}{z^{p-1}} * q(z)} \quad (z \in U). \end{aligned} \tag{2.21}$$

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Now, by using (2.16), (2.18), (2.20) and (2.21), an application of Lemma 1.2 leads to

$$\frac{L_p(a+1,c)(f*g)(z)}{L_p(a,c)(f*g)(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

This shows that $(f*g)(z) \in Q_p(a,c;h)$. □

For $\alpha = 0$ and $\alpha = \frac{1}{2}$, Theorem 2.7 reduces to

Corollary 2.8. *Let $h(z) \in P$ and $g(z) \in A_p$ satisfy either*

(i) $\frac{g(z)}{z^{p-1}}$ *is convex univalent in U and*

$$\operatorname{Re} h(z) > p-1 \quad (z \in U)$$

or

(ii) $\frac{g(z)}{z^{p-1}} \in S^*(\frac{1}{2})$ *and*

$$\operatorname{Re} h(z) > p - \frac{1}{2} \quad (z \in U).$$

If $f(z) \in Q_p(a,c;h)$, then

$$(f*g)(z) \in Q_p(a,c;h).$$

Theorem 2.9. *Let $h(z) \in P$ and*

$$\operatorname{Re} h(z) > -\operatorname{Re}\lambda \quad (z \in U), \tag{2.22}$$

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where λ is a complex number such that $\operatorname{Re}\lambda > -p$. If $f(z) \in Q_p(a,c;h)$, then the function

$$g(z) = \frac{\lambda+p}{z^\lambda}\int_0^z t^{\lambda-1}f(t)dt \tag{2.23}$$

is also in the class $Q_p(a,c;h)$.

Proof. For $f(z) \in A_p$ and $\operatorname{Re}\lambda > -p$, it follows from (1.7) and (2.23) that $g(z) \in A_p$ and

$$\begin{aligned}(\lambda+p)L_p(a,c)f(z) &= \lambda L_p(a,c)g(z) + z(L_p(a,c)g(z))' \\ &= aL_p(a+1,c)g(z) + (\lambda+p-a)L_p(a,c)g(z).\end{aligned} \tag{2.24}$$

If we let

$$q(z) = \frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}, \tag{2.25}$$

then (2.24) and (2.25) lead to

$$aq(z)+\lambda+p-a = (\lambda+p)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}. \tag{2.26}$$

Differentiating both sides of (2.26) logarithmically and using (1.7) and (2.25), we obtain

$$\begin{aligned}\frac{zq'(z)}{aq(z)+\lambda+p-a} &= \frac{1}{a}\left(\frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} - \frac{z(L_p(a,c)g(z))'}{L_p(a,c)g(z)}\right) \\ &= \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - q(z).\end{aligned} \tag{2.27}$$

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Let $f(z) \in Q_p(a, c; h)$. Then it follows from (2.27) that

$$q(z) + \frac{zq'(z)}{aq(z) + \lambda + p - a} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}. \tag{2.28}$$

Also, in view of (2.22), we have

$$\operatorname{Re}\left\{a\left(1 - \frac{p}{a} + \frac{h(z)}{a}\right) + \lambda + p - a\right\} = \operatorname{Re} h(z) + \operatorname{Re}\lambda > 0 \quad (z \in U). \tag{2.29}$$

Therefore, it follows from (2.28), (2.29) and Lemma 1.3 that

$$q(z) \prec 1 - \frac{p}{a} + \frac{h(z)}{a}.$$

This proves that $g(z) \in Q_p(a, c; h)$. □

From Theorem 2.9 we have the following corollaries.

Corollary 2.10. *Let $h(z)$ be defined as in Corollary 2.2. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re}\lambda \geq 1 - p - \left(\frac{1-A}{1-B}\right)^{\gamma} \quad (0 < \gamma \leq 1; -1 \leq B < A \leq 1),$$

then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.

In the special case when $\gamma = 1$, Corollary 2.10 was obtained by Liu and Owa [5, Theorem 2] using Jack's Lemma.

Corollary 2.11. *Let $h(z)$ be defined as in Corollary 2.4. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re}\lambda \geq \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{\gamma+1}{\gamma+k}\right) \delta^k - p \quad (0 < \delta \leq 1; \gamma \geq 0),$$

then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.

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Corollary 2.12. *Let $h(z)$ be defined as in Corollary 2.5. If $f(z) \in Q_p(a, c; h)$ and*

$$\operatorname{Re}\lambda \geq \frac{8}{\pi^2}(\arctan\sqrt{\gamma})^2 - p \quad (0 < \gamma \leq 1),$$

then the function $g(z)$ given by (2.23) is also in the class $Q_p(a, c; h)$.

Theorem 2.13. *Let $h(z) \in P$ and*

$$\operatorname{Re} h(z) > -\frac{\operatorname{Re}\lambda}{\beta} \quad (z \in U), \tag{2.30}$$

where $\beta > 0$ and λ is a complex number such that $\operatorname{Re}\lambda > -p\beta$. If $f(z) \in Q_p(a, c; h)$, then the function $g(z) \in A_p$ defined by

$$L_p(a, c)g(z) = \left(\frac{\lambda + p\beta}{z^\lambda}\int_0^z t^{\lambda-1}\left(L_p(a, c)f(t)\right)^\beta dt\right)^{\frac{1}{\beta}} \tag{2.31}$$

is also in the class $Q_p(a, c; h)$.

Proof. Let $f(z) \in Q_p(a, c; h)$. From the definition of $g(z)$ we have

$$z^\lambda(L_p(a, c)g(z))^\beta = (\lambda + p\beta)\int_0^z t^{\lambda-1}(L_p(a, c)f(t))^\beta dt. \tag{2.32}$$

Differentiating both sides of (2.32) logarithmically and using (1.7), we get

$$\lambda + \beta(aq(z) + p - a) = (\lambda + p\beta)\left(\frac{L_p(a, c)f(z)}{L_p(a, c)g(z)}\right)^\beta, \tag{2.33}$$

where

$$q(z) = \frac{L_p(a + 1, c)g(z)}{L_p(a, c)g(z)}. \tag{2.34}$$

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Also, differentiating both sides of (2.33) logarithmically and using (1.7), we arrive at

$$q(z) + \frac{zq'(z)}{a\beta q(z) + \lambda + \beta(p-a)} = \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a}. \tag{2.35}$$

Noting that (2.30) and $\beta > 0$, we see that

$$\operatorname{Re}\left\{a\beta\left(1 - \frac{p}{a} + \frac{h(z)}{a}\right) + \lambda + \beta(p-a)\right\} \\ = \beta \operatorname{Re} h(z) + \operatorname{Re}\lambda > 0 \quad (z \in U). \tag{2.36}$$

Now, in view of (2.34), (2.35) and (2.36), an application of Lemma 1.3 yields

$$\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)} \prec 1 - \frac{p}{a} + \frac{h(z)}{a},$$

that is, $g(z) \in Q_p(a,c;h)$. □

Corollary 2.14. *Let $h(z)$ be defined as in Corollary 2.2. If $f(z) \in Q_p(a,c;h)$ and*

$$\operatorname{Re}\lambda \geq \beta\left(1 - p - \left(\frac{1-A}{1-B}\right)^{\gamma}\right) \quad (0 < \gamma \leq 1; -1 \leq B < A \leq 1; \beta > 0),$$

then the function $g(z) \in A_p$ defined by (2.31) is also in the class $Q_p(a,c;h)$.

Analytic and Multivalent Functions Defined by Linear Operators
Ding-Gong Yang, N-eng Xu and Shigeyoshi Owa
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