Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 4, Article 144, 2006

FURTHER DEVELOPMENT OF QI-TYPE INTEGRAL INEQUALITY

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Received 27 July, 2006; accepted 22 November, 2006
Communicated by S.S. Dragomir

ABSTRACT. We give some further answers to the open problem posed in the article [Feng Qi, Several integral inequalities, *J. Inequal. Pure and Appl. Math.*, **1**(2) (2000), Art. 19. (http://jipam.vu.edu.au/article.php?sid=113]).] Being Qi's inequality of moment type, we consider the moments of uniformly distributed random variables and construct certain suitable probability measures to solve the posed problem. Moreover, reverse inequality to Qi's and other related results are deduced as well.

Key words and phrases: Hölder inequality, Integral inequality, Jensen's inequality, Nehari inequality, Qi-type inequality.

2000 *Mathematics Subject Classification.* 26D15.

1. INTRODUCTION

The following problem was posed by F. Qi in his article: *"Under what condition does the inequality*

$$\int_a^b [f(x)]^t \, dx \geq \left(\int_a^b f(x) dx \right)^{t-1} \tag{1.1}$$

hold for $t > 1$?", [9].

There are numerous answers and extension results to this open problem [1, 2, 3, 4, 5, 7, 8, 10, 11, 12]. These results were obtained via different approaches, such as, e.g. Jensen's inequality, the convexity method [12]; functional inequalities in abstract spaces [1, 2]; probability measures techniques [4]; Hölder inequality and its reversed variants [2, 8]; analytical methods [7, 11] and Cauchy's mean value theorem [3, 10].

Here and in what follows we write $X \sim \mathcal{U}[a; b]$ for the random variable (r.v.) X which possesses uniform distribution on the support interval $[a, b]$, i.e., the probability density function

ISSN (electronic): 1443-5756

The author is indebted to Professor Qi and the anonymous referees for their many helpful comments and for many valuable additions to the list of references.

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of X is equal to $(b-a)^{-1}$, $x \in [a,b]$ and zero elsewhere. Accordingly, let us denote $\mathbb{E}Z$ the mathematical expectation of r.v. Z.

In this paper we obtain generalizations of (1.1) and extend some results of [1, 2, 4, 5, 7, 8, 9, 11, 12] using moment properties of uniformly distributed r.v.s and applying some moment inequalities of suitably constructed probability measures. To do this we introduce the extension of (1.1) by Pogány, [8]: *"Under what conditions does the inequality*

$$\int_a^b \big[f(x)\big]^\alpha dx \geq \left(\int_a^b f(x)dx\right)^\beta, \quad (\alpha,\beta>0) \tag{1.2}$$

hold?"

Indeed, specifying $\alpha=\beta+1=t>0$ in (1.2) we arrive at (1.1).

We will consider moment type inequalities for a function of the r.v. $X \sim \mathcal{U}[a,b]$. In Section 2 we obtain results concerning the direct inequality (1.2) by taking the probability distribution function for uniform distribution. In Section 3 we derive some inequalities reversed to (1.2) relaxing the conditions upon f given in [8]. Finally, in Section 4 bounded and semi-bounded integrands will be treated by constructing suitable probability measures for arriving at answers to (1.2).

2. Direct Inequality

In this section we delineate two important cases for considering (1.2). First, let $\alpha > \max\{1,\beta\}$, then we take $\alpha>0, \beta>1$.

2.1. The case $\alpha > \max\{1,\beta\}$.

Firstly we introduce the following auxiliary inequality which will be frequently needed in the sequel:

$$\left(\int_a^b f(x)dx\right)^{\beta-\alpha} \leq (b-a)^{1-\alpha}. \tag{2.1}$$

Now, looking for the widest possible class of integrands f such that (1.2) remains valid under the constraint $\alpha > \max\{1,\beta\}$, we obtain the following result.

Theorem 2.1. *Let $f \in C[a,b]$, f^α be integrable on $[a,b]$. When one of the following two conditions holds*

(R_1) $((2.1)\ \&\ f \geq 0,\ \beta>0)$;
(R_2) $((2.1)\ \&\ \beta \geq 0,\ \alpha = 2k/j > 1,\ j,k \in \mathbb{N})$;

then the inequality (1.2) also holds.

Proof. Let $X \sim \mathcal{U}[a,b]$. Then it is obvious that

$$\int_a^b f(x)dx = (b-a)\mathbb{E}f(X) \quad \text{and} \quad \int_a^b \big[f(x)\big]^\alpha dx = (b-a)\mathbb{E}\big[f(X)\big]^\alpha. \tag{2.2}$$

Thus it is sufficient to show

$$(b-a)\mathbb{E}\big[f(X)\big]^\alpha \geq \Big[(b-a)\mathbb{E}f(X)\Big]^\beta. \tag{2.3}$$

Indeed, bearing in mind (R_1), by Jensen's inequality we conclude

$$\begin{aligned}\Big[(b-a)\mathbb{E}f(X)\Big]^\beta &= (b-a)^\beta\Big[\mathbb{E}f(X)\Big]^\alpha\big[\mathbb{E}f(X)\big]^{\beta-\alpha} \\ &\leq (b-a)^\beta\mathbb{E}\big[f(X)\big]^\alpha\big[\mathbb{E}f(X)\big]^{\beta-\alpha} \\ &\leq (b-a)\mathbb{E}\big[f(X)\big]^\alpha.\end{aligned} \tag{2.4}$$

The proof under (R_1) is finished. To apply the condition (R_2) it is enough to notice that x^α is convex on $\mathbb{R}$ for all $\alpha = 2k/j > 1$, j, k being positive integers. These considerations complete the proof of the theorem. □

Remark 2.2.

$(\mathbf{A})$ Yu and Qi [12] proved the inequality (1.1) for $f \in C[a, b]$ under (R_1). Then Mazouzi and Qi [5] proved (1.1) by a functional inequality, which reads as follows,

$$|f(x)| \geq k(x), \quad \text{a.e.} \quad x \in [a, b] \quad \text{and} \quad (b-a)^{\frac{\alpha-1}{\alpha-\beta}} \leq \int_a^b k(x)dx < \infty.$$

$(\mathbf{B})$ Condition (R_2) ensures the validity of the first inequality in (2.4). Assuming only (2.1) without the condition $\alpha = 2k/j > 1$, $j, k \in \mathbb{N}$, the first inequality in (2.4) could be false. Indeed, the r.v. $\xi \sim \mathcal{U}[-c, 0]$, $c > 0$ presents a simple counterexample to the statement

$$\big[\mathbb{E}f(X)\big]^\alpha \leq \mathbb{E}\big[f(X)\big]^\alpha,$$

since

$$\big[\mathbb{E}\xi\big]^{2\kappa-1} = -\left(\frac{c}{2}\right)^{2\kappa-1} \geq -\frac{c^{2\kappa-1}}{2\kappa} = \mathbb{E}\xi^{2\kappa-1}, \qquad \kappa \in \mathbb{N}.$$

2.2. **The case** $\alpha > 0, \beta > 1$**.** In this case we will need the help of an auxiliary result, which we clearly deduce by the Hölder inequality.

Lemma 2.3. *Let Z, Y be two random variables with $Z \geq 0$, $Y \geq 0$, $Z/Y \geq 0$ a.e. and $\mathbb{E}(Z/Y)^{rp} \leq K$ for some constant K, $1/p + 1/q = 1$. Then*

$$\mathbb{E}Z^r \leq \big[\mathbb{E}(Z/Y)^{rp}\big]^{1/p} \big[\mathbb{E}Y^{rq}\big]^{1/q} \leq K^{1/p}\big[\mathbb{E}Y^{rq}\big]^{1/q}, \tag{2.5}$$

where $r > 0$.

Theorem 2.4. *Suppose f is a positive continuous function on $[a, b]$, f^γ is integrable on $[a, b]$, where $\gamma := \max\{1, \alpha\}$, and for $\alpha > 0$, $\beta > 1$, the following condition is satisfied*

$$\int_a^b \big[f(x)\big]^{(\beta-\alpha)/(\beta-1)}dx \leq 1. \tag{2.6}$$

Specifically, for $\alpha > \beta$, letting $f(x) \geq m > 0$ and $(b-a)/m^{\frac{\alpha-\beta}{\beta-1}} \leq 1$, then the inequality (1.2) holds true.

Proof. Let the r.v. $X \sim \mathcal{U}[a, b]$. Thus, (2.2) holds. Therefore it is enough to prove that

$$(b-a)\mathbb{E}\big[f(X)\big]^\alpha \geq \Big[(b-a)\mathbb{E}f(X)\Big]^\beta.$$

Let $q = \beta > 1$, $p = \frac{\beta}{\beta-1}$, $Z^r = f(X)$ and $Y^{r\beta} = \big[f(X)\big]^\alpha$, in the formula of Lemma 2.3. Then $(Z/Y)^r = \big[f(X)\big]^{1-\alpha/\beta}$ readily follows, and consequently

$$\begin{aligned}
\big[(b-a)\mathbb{E}f(X)\big]^\beta &\leq \Big[(b-a)\Big(\mathbb{E}\big[f(X)\big]^{p-p\alpha/\beta}\Big)^{1/p}\Big(\mathbb{E}\big[f(X)\big]^\alpha\Big)^{1/\beta}\Big]^\beta \\
&= (b-a)^{\beta-1}\Big(\mathbb{E}\big[f(X)\big]^{p-p\alpha/\beta}\Big)^{\beta/p}(b-a)\Big(\mathbb{E}\big[f(X)\big]^\alpha\Big) \\
&= (b-a)^{\beta-1}\Big(\mathbb{E}\big[f(X)\big]^{(\beta-\alpha)/(\beta-1)}\Big)^{\beta-1}(b-a)\Big(\mathbb{E}\big[f(X)\big]^\alpha\Big) \\
&= \left(\int_a^b \big[f(x)\big]^{(\beta-\alpha)/(\beta-1)}dx\right)^{\beta-1}(b-a)\Big(\mathbb{E}\big[f(X)\big]^\alpha\Big).
\end{aligned}$$

Now, by (2.6) we conclude the desired inequality (1.2). □

Remark 2.5. In fact, we do not need the condition $\alpha > \beta$, since supposing the converse $\alpha < \beta$ and $(\beta - \alpha)/(\beta - 1) < 1$, then the condition (2.6) can be replaced by the following condition (from using x^γ, $0 < \gamma < 1$, concave):

$$\int_a^b f(x)dx \le (b-a)^{\frac{1-\alpha}{\beta-\alpha}},$$

which is easier to check.

3. Reverse Qi-Type inequality

In this section, we mainly discuss reverse inequalities of the Qi-type inequality (1.2), and at the same time we improve the results of Pogány [8]. For this purpose we list another auxiliary inequality derived by Nehari [6], which is a reverse of the celebrated Hölder inequality.

Lemma 3.1 (Nehari Inequality)**.** *Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p,q > 0$ such that $1/p + 1/q = 1$, we have*

$$\left(\int_a^b \left[f(x)\right]^p dx\right)^{\frac{1}{p}} \left(\int_a^b \left[g(x)\right]^q dx\right)^{\frac{1}{q}} \le N(p,q) \int_a^b f(x)g(x)dx, \tag{3.1}$$

where

$$N(p,q) = \frac{6}{(1+p)^{1/p}(1+q)^{1/q}}. \tag{3.2}$$

Theorem 3.2. *Let $f(x)$ be nonnegative, concave and integrable on $[a,b]$, $\beta > 0$ and $\max\{\beta, 1\} < \alpha$. Assume*

$$\int_a^b f(x)dx \le (b-a)\left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^\alpha(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}}\right)^{\frac{1}{\alpha-\beta}}. \tag{3.3}$$

Then the reverse inequality to (1.2), i.e.,

$$\int_a^b \left[f(x)\right]^\alpha dx \le \left[\int_a^b f(x)dx\right]^\beta \tag{3.4}$$

holds true.

Proof. Let $X \sim \mathcal{U}[a,b]$. As (2.2) is valid, we are confronted with the proof of

$$(b-a)\mathbb{E}\big[f(X)\big]^\alpha \le \Big[(b-a)\mathbb{E}f(X)\Big]^\beta. \tag{3.5}$$

The Nehari inequality (3.1) can be written in an equivalent form as

$$(b-a)\big(\mathbb{E}\big[f(X)\big]^p\big)^{1/p}\big(\mathbb{E}\big[g(X)\big]^q\big)^{1/q} \le N(p,q)\int_a^b f(x)g(x)dx. \tag{3.6}$$

Taking $g \equiv 1$, $p = \alpha$, then (3.6) becomes

$$\Big(\mathbb{E}\big[f(X)\big]^\alpha\Big)^{1/\alpha} \le N\left(\alpha, \frac{\alpha}{\alpha-1}\right)\mathbb{E}f(X). \tag{3.7}$$

Thus by (3.7) and (3.3), we deduce

$$\begin{aligned}(b-a)\mathbb{E}\big[f(X)\big]^\alpha &\le (b-a)N^\alpha\left(\alpha, \frac{\alpha}{\alpha-1}\right)\big[\mathbb{E}f(X)\big]^\alpha \\ &= (b-a)^{1-\beta}N^\alpha\left(\alpha, \frac{\alpha}{\alpha-1}\right)\big[\mathbb{E}f(X)\big]^{\alpha-\beta}\big[(b-a)\mathbb{E}f(X)\big]^\beta \\ &= \big[(b-a)\mathbb{E}f(X)\big]^\beta.\end{aligned}$$

This ends the proof of (3.4). □

Remark 3.3.

(**A**) Pogány [8] derived (3.4) for all f such that

$$0 \le f(x) \le \left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^\alpha(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad x \in [a,b]. \tag{3.8}$$

It is easy to see that our condition (3.3) relaxes (3.8).

(**B**) Csiszár and Móri [4] improved the results of Pogány [8] and obtained the inequality (3.4) under the following condition

$$f(x) \le \left(\frac{1+\alpha}{2^\alpha(b-a)^{1-\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad x \in [a,b]. \tag{3.9}$$

The last constraint is obviously weaker than (3.8), but does not cover our integral condition (3.3).

4. SOLVING (1.2) BY CONSTRUCTING SUITABLE PROBABILITY MEASURES

In this section we consider bounded and/or semi-bounded functions, and construct convenient probability measures, different to $\mathcal{U}[a,b]$. Then, considering certain relations between its moments, we derive new Qi-type inequality results.

Theorem 4.1. *Assume that* $0 < m \le f \le M < \infty$, *and for* $\alpha > \beta > 1$,

$$\frac{m^{\alpha-1}}{M^{\beta-1}(b-a)^{\beta-1}} \ge 1, \tag{4.1}$$

then

$$\int_a^b \big[f(x)\big]^\alpha dx \ge \left[\int_a^b f(x)dx \right]^\beta.$$

Moreover, the reverse inequality to (1.2) is valid when

$$\frac{M^{\alpha-1}}{m^{\beta-1}(b-a)^{\beta-1}} \le 1. \tag{4.2}$$

Proof. Define

$$\mu(t) = \int_a^t \frac{f(x)}{\int_a^b f(x)dx} dx, \quad t \in [a,b].$$

It is easy to see that $\mu(\cdot)$ orders a probability measure on $[a,b]$ and the following implications follow

$$\begin{aligned} \frac{\int_a^b \big[f(x)\big]^\alpha dx}{\big[\int_a^b f(x)dx\big]^\beta} &= \int_a^b \big[f(x)\big]^{\alpha-1} \frac{f(x)}{\int_a^b f(x)dx} dx \frac{1}{\big[\int_a^b f(x)dx\big]^{\beta-1}} \\ &= \frac{\int_a^b \big[f(x)\big]^{\alpha-1} \mu(dx)}{\big[\int_a^b f(x)dx\big]^{\beta-1}} \\ &\ge \frac{m^{\alpha-1}}{M^{\beta-1}(b-a)^{\beta-1}}. \end{aligned} \tag{4.3}$$

The remaining part of the proof is straightforward. □

Remark 4.2.

$(\mathbf{A})$ The direct use of the assumption $m \le f(x) \le M$, $m > 0$ in the sharpness evaluation of (4.1) results in

$$\frac{\int_a^b \left[f(x)\right]^\alpha dx}{\left[\int_a^b f(x)dx\right]^\beta} \ge \frac{m^\alpha}{M^\beta (b-a)^{\beta-1}} =: \mathfrak{M}_1. \tag{4.4}$$

For our purposes we need the case $\mathfrak{M}_1 \ge 1$. However, it is easy to check that

$$\mathfrak{M}_1 \le \frac{m^{\alpha-1}}{M^{\beta-1}(b-a)^{\beta-1}};$$

hence, (4.1) generalizes the simplest possible $\mathfrak{M}_1 \ge 1$.

$(\mathbf{B})$ By similar arguments,

$$\mathfrak{M}_2 := \frac{M^\alpha}{m^\beta (b-a)^{\beta-1}} \le 1$$

implies (4.2), so, when the considered integrand functions are bounded and positive, the settings of Theorem 4.1 are optimal.

Corollary 4.3. *Assume that* $0 < m \le f \le M < \infty$, *and for* $0 < \beta < \alpha < 1$,

$$\frac{M^{\alpha-1}}{m^{\beta-1}(b-a)^{\beta-1}} \ge 1, \tag{4.5}$$

then the validity of the inequality (1.2) is confirmed.

Moreover, for $0 < \beta < \alpha < 1$, *if* $0 < m \le f \le M < \infty$ *and*

$$\frac{m^{\alpha-1}}{M^{\beta-1}(b-a)^{\beta-1}} \le 1, \tag{4.6}$$

there follows the inequality which is reversed to (1.2).

Corollary 4.4. *Assume that* $0 < m \le f < \infty$, $0 < \beta < 1 < \alpha$, *let* f^α *be integrable on* $[a, b]$ *and*

$$\mathfrak{N}_1 := \frac{m^{\alpha-\beta}}{(b-a)^{\beta-1}} \ge 1. \tag{4.7}$$

Then (1.2) follows. Otherwise, when $0 < \beta < \alpha < 1$, $0 < f \le M < \infty$ *and*

$$\mathfrak{N}_2 := \frac{M^{\alpha-\beta}}{(b-a)^{\beta-1}} \le 1, \tag{4.8}$$

the reverse inequality to (1.2) is deduced.

Finally, let us construct an another probability measure

$$\mu_\beta(x) := \frac{\int_a^x [f(t)]^\beta dt}{\int_a^b [f(t)]^\beta dt}, \qquad x \in [a, b], \quad \beta \ne 1. \tag{4.9}$$

Taking into account the previous procedure for getting Qi-type inequalities and their reversed variants, we arrive at the following results.

Theorem 4.5. *Assume* $0 < m \le f < \infty$, *let* f^α *be integrable on* $[a, b]$ *and for* $\alpha > \beta > 1$, *let us suppose* $\mathfrak{N}_1 \ge 1$. *Then we have the inequality (1.2).*

In addition, for $0 < \beta < 1$, $\alpha > \beta$, $0 < f \le M < \infty$ *as* $x \in [a, b]$ *and* $\mathfrak{N}_2 \le 1$, *then the reverse inequality to (1.2) holds true.*

Proof. Let us consider the probability measure $\mu_\beta(x)$, $x \in [a,b]$, $\beta > 1$:

$$\begin{aligned}\frac{\int_a^b \big[f(x)\big]^\alpha dx}{\big[\int_a^b f(x)dx\big]^\beta} &= \frac{\int_a^b \big[f(x)\big]^\alpha dx}{\big[(b-a)\mathbb{E}f(X)\big]^\beta} \\ &\geq \int_a^b \big[f(x)\big]^{\alpha-\beta} \frac{[f(x)]^\beta}{(b-a)^{\beta-1}\int_a^b [f(x)]^\beta dx} dx \\ &= (b-a)^{1-\beta} \int_a^b \big[f(x)\big]^{\alpha-\beta} \mu(dx) \\ &\geq (b-a)^{1-\beta} m^{\alpha-\beta} = \mathfrak{N}_1.\end{aligned}$$

This is equivalent to the assertion of Theorem 4.5.

The proof of the second case we leave to the interested reader. □

By a similar proof procedure as the previous theorem, we obtain the following interesting result.

Theorem 4.6. *Assume that* $0 < f \leq M < \infty$*, let* f^α *be integrable on* $[a,b]$ *and for* $\beta > \max\{1,\alpha\}$*,* $\alpha > 0$*, we let* $\mathfrak{N}_2 \geq 1$*. Then we have the inequality (1.2).*

Additionally, for $0 < \alpha < \beta < 1$*,* $0 < m \leq f < \infty$ *as* $x \in [a,b]$ *and* $\mathfrak{N}_1 \leq 1$*, then the reverse inequality to (1.2) holds true.*

Because of the similarity of the proofs of last two theorems the proof of the last one is omitted.

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