ABOUT A CLASS OF LINEAR POSITIVE OPERATORS OBTAINED BY CHOOSING THE NODES

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Received: 15 June, 2007

Accepted: 18 March, 2009

Communicated by: S.S. Dragomir

2000 AMS Sub. Class.: 41A10, 41A25, 41A35, 41A36.

Key words: Linear positive operators, convergence theorem, the first order modulus of smoothness, approximation theorem.

Abstract: In this paper we consider the given linear positive operators $(L_m)_{m\geq 1}$ and with their help, we construct linear positive operators $(\mathcal{K}_m)_{m\geq 1}$. We study the convergence, the evaluation for the rate of convergence in terms of the first modulus of smoothness for the operators $(\mathcal{K}_m)_{m\geq 1}$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

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journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

In this section, we recall some notions and operators which we will use in this article.

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let $B_m : C([0,1]) \to C([0,1])$ be Bernstein operators, defined for any function $f \in C([0,1])$ by

$$(1.1)\qquad (B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$(1.2)\qquad p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k},$$

for any $x \in [0,1]$ and any $k \in \{0,1,\dots,m\}$ (see [5] or [24]). For the following construction, see [15]. Define the natural number m_0 by

$$(1.3)\qquad m_0 = \begin{cases} \max(1, -[\beta]), & \text{if} \quad \beta \in \mathbb{R} - \mathbb{Z}; \\ \max(1, 1-\beta), & \text{if} \quad \beta \in \mathbb{Z}, \end{cases}$$

where $[x]$, $\{x\}$ denote the integer and fractional parts respectively of a real number x.

For the real number β, we have that

$$(1.4)\qquad m + \beta \ge \gamma_\beta$$

for any natural number m, $m \ge m_0$, where

$$(1.5)\qquad \gamma_\beta = m_0 + \beta = \begin{cases} \max\left(1+\beta, \{\beta\}\right), & \text{if} \quad \beta \in \mathbb{R} - \mathbb{Z}; \\ \max(1+\beta, 1), & \text{if} \quad \beta \in \mathbb{Z}. \end{cases}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

For the real numbers α, β, $\alpha \geq 0$, we note

$$\mu^{(\alpha,\beta)} = \begin{cases} 1, & \text{if} \quad \alpha \leq \beta; \\ 1 + \frac{\alpha-\beta}{\gamma_\beta}, & \text{if} \quad \alpha > \beta. \end{cases} \tag{1.6}$$

For the real numbers α and β, $\alpha \geq 0$, we have that $1 \leq \mu^{(\alpha,\beta)}$ and

$$0 \leq \frac{k+\alpha}{m+\beta} \leq \mu^{(\alpha,\beta)} \tag{1.7}$$

for any natural number m, $m \geq m_0$ and for any $k \in \{0, 1, \ldots, m\}$.

For the real numbers α and β, $\alpha \geq 0$, m_0 and $\mu^{(\alpha,\beta)}$ defined by (1.3) – (1.6), let the operators $P_m^{(\alpha,\beta)} : C\left([0, \mu^{(\alpha,\beta)}]\right) \to C\left([0,1]\right)$, defined for any function $f \in C\left([0, \mu^{(\alpha,\beta)}]\right)$ by

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right), \tag{1.8}$$

for any natural number $m, m \geq m_0$ and for any $x \in [0,1]$. These operators are called Stancu operators, and were introduced and studied in 1969 by D.D. Stancu in the paper [23]. In [23], the domain of definition of Stancu's operators is $C([0,1])$ and the numbers α and β verify the condition $0 \leq \alpha \leq \beta$.

In 1980, G. Bleimann, P. L. Butzer and L. Hahn introduced in [4] a sequence of linear positive operators $(L_m)_{m \geq 1}$, $L_m : C_B([0,\infty)) \to C_B([0,\infty))$, defined for any function $f \in C_B([0,\infty))$ by

$$(L_m f)(x) = \frac{1}{(1+x)^m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{k}{m+1-k}\right), \tag{1.9}$$

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for any $x \in [0,\infty)$ and any $m \in \mathbb{N}$, where $C_B([0,\infty)) = \{f \mid f : [0,\infty) \to \mathbb{R}, f$ is bounded and continuous on $[0,\infty)\}$.

For $m \in \mathbb{N}$, consider the operators $S_m : C_2([0,\infty)) \to C([0,\infty))$ defined for any function $f \in C_2([0,\infty))$ by

$$(S_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{k}{m}\right), \tag{1.10}$$

for any $x \in [0,\infty)$, where

$$C_2([0,\infty)) = \left\{ f \in C([0,\infty)) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \text{ exists and is finite} \right\}.$$

The operators $(S_m)_{m\geq 1}$ are called Mirakjan-Favard-Szász operators and were introduced in 1941 by G. M. Mirakjan in [12].

They were intensively studied by J. Favard in 1944 in [8] and O. Szász in 1950 in [25].

For $m \in \mathbb{N}$, the operator $V_m : C_2([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C_2([0,\infty))$ by

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right), \tag{1.11}$$

for any $x \in [0,\infty)$.

The operators $(V_m)_{m\geq 1}$ are named Baskakov operators and they were introduced in 1957 by V. A. Baskakov in [2].

W. Meyer-König and K. Zeller have introduced in [11] a sequence of linear and positive operators. After a slight adjustment, given by E.W. Cheney and A. Sharma

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in [6], these operators take the form $Z_m : B([0,1)) \to C([0,1))$, defined for any function $f \in B([0,1))$ by

$$(1.12) \qquad (Z_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k}{m+k}\right),$$

for any $m \in \mathbb{N}$ and for any $x \in [0,1)$.

These operators are called the Meyer-König and Zeller operators.

Observe that $Z_m : C([0,1]) \to C([0,1])$, $m \in \mathbb{N}$.

In [10], M. Ismail and C.P. May consider the operators $(R_m)_{m\geq 1}$.

For $m \in \mathbb{N}$, $R_m : C([0,\infty)) \to C([0,\infty))$ is defined for any function $f \in C([0,\infty))$ by

$$(1.13) \qquad (R_m f)(x) = e^{-\frac{mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{k}{m}\right)$$

for any $x \in [0,\infty)$.

We consider $I \subset \mathbb{R}$, I an interval and we shall use the following function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f \mid f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$.

If $f \in B(I)$, then the first order modulus of smoothness of f is the function $\omega(f;\cdot) : [0,\infty) \to \mathbb{R}$ defined for any $\delta \geq 0$ by

$$(1.14) \qquad \omega(f;\delta) = \sup\{|f(x') - f(x'')| : x', x'' \in I, |x' - x''| \leq \delta\}.$$

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2. Preliminaries

For the following construction and result see [16] and [18], where $p_m = m$ for any $m \in \mathbb{N}$ or $p_m = \infty$ for any $m \in \mathbb{N}$. Let $I, J \subset [0, \infty)$ be intervals with $I \cap J \neq \emptyset$. For any $m \in \mathbb{N}$ and $k \in \{0, 1, ..., p_m\} \cap \mathbb{N}_0$ consider the nodes $x_{m,k} \in I$ and the functions $\varphi_{m,k} : J \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in J$. Let $E(I)$ and $F(J)$ be subsets of the set of real functions defined on I, respectively J so that the sum

$$\sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k})$$

exists for any $f \in E(I)$, $x \in J$ and $m \in \mathbb{N}$. For any $x \in I$ consider the functions $\psi_x : I \to \mathbb{R}$, $\psi_x(t) = t - x$ and $e_i : I \to \mathbb{R}$, $e_i(t) = t^i$ for any $t \in I$, $i \in \{0, 1, 2\}$. In the following, we suppose that for any $x \in I$ we have $\psi_x \in E(I)$ and $e_i \in E(I)$, $i \in \{0, 1, 2\}$.

For $m \in \mathbb{N}$, let the given operator $L_m : E(I) \to F(J)$ defined by

$$(L_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(x_{m,k}) \tag{2.1}$$

with the property that the convergence

$$\lim_{m\to\infty} (L_m f)(x) = f(x) \tag{2.2}$$

is uniform on any compact $K \subset I \cap J$, for any $f \in E(I) \cap C(I)$.

Remark 1. From (2.2), for the operators $(L_m)_{m\geq 1}$ we have that the following convergences

$$\lim_{m\to\infty} (L_m e_i)(x) = e_i(x), \tag{2.3}$$

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$i \in \{0, 1, 2\}$ and

$$\lim_{m\to\infty} (L_m \psi_x^2)(x) = 0 \tag{2.4}$$

are uniform on any compact $K \subset I \cap J$.

Remark 2. From Remark 1 it results that for any compact $K \subset I \cap J$ the sequences $(u_m(K))_{m\geq 1}$, $(v_m(K))_{m\geq 1}$, $(w_m(K))_{m\geq 1}$ depending on K exist, so that the convergences

$$\lim_{m\to\infty} u_m(K) = \lim_{m\to\infty} v_m(K) = \lim_{m\to\infty} w_m(K) = 0 \tag{2.5}$$

are uniform on K and

$$|(L_m e_0)(x) - 1| \leq u_m(K), \tag{2.6}$$

$$|(L_m e_1)(x) - x| \leq v_m(K), \tag{2.7}$$

$$(L_m \psi_x^2)(x) \leq w_m(K), \tag{2.8}$$

for any $x \in K$ and any $m \in \mathbb{N}$.

In the following, for $m \in \mathbb{N}$ and $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ we consider the nodes $y_{m,k} \in I$ so that

$$\alpha_m = \sup_{k\in\{0,1,\dots,p_m\}\cap\mathbb{N}_0} |x_{m,k} - y_{m,k}| < \infty \tag{2.9}$$

for any $m \in \mathbb{N}$ and

$$\lim_{m\to\infty} \alpha_m = 0. \tag{2.10}$$

For $m \in \mathbb{N}$ and $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ we note that $\alpha_{m,k} = x_{m,k} - y_{m,k}$.

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Definition 2.1. *For $m \in \mathbb{N}$, define the operator $\mathcal{K}_m : E(I) \to F(J)$ by*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{p_m} \varphi_{m,k}(x) f(y_{m,k}), \tag{2.11}$$

for any $x \in I$ and any $f \in E(I)$.

Remark 3. Similar ideas to the construction above can be found in the recent papers [9] and [13].

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3. Main Results

In this section, we study the operators defined by (2.11).

Theorem 3.1. *For any $f \in E(I) \cap C(I)$ we have that the convergence*

$$\lim_{m\to\infty} (\mathcal{K}_m f)(x) = f(x) \tag{3.1}$$

is uniform on any compact $K \subset I \cap J$.

Proof. For $x \in K$ and $m \in \mathbb{N}$ we have that

$$\begin{aligned}
(\mathcal{K}_m \psi_x^2)(x) &= (\mathcal{K}_m e_2)(x) - 2x(\mathcal{K}_m e_1)(x) + x^2(\mathcal{K}_m e_0)(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) y_{m,k} + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \alpha_{m,k})^2 \\
&\qquad - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x)(x_{m,k} - \alpha_{m,k}) + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) \\
&= \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k}^2 - 2 \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k} \alpha_{m,k} \\
&\qquad + \sum_{k=0}^{p_m} \varphi_{m,k}(x) \alpha_{m,k}^2 - 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) x_{m,k} \\
&\qquad + 2x \sum_{k=0}^{p_m} \varphi_{m,k}(x) \alpha_{m,k} + x^2 \sum_{k=0}^{p_m} \varphi_{m,k}(x)
\end{aligned}$$

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$$\leq (L_m\psi_x^2)(x) + 2\alpha_m(L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x).$$

Taking Remark 1 and Remark 2 into account, it results that (3.1) holds. □

Theorem 3.2. *If $f \in E(I \cap J) \cap C(I \cap J)$, then for any $x \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$, we have that*

$$(3.2)\quad |(\mathcal{K}_m f)(x) - f(x)| \leq |f(x)|\,|(L_m e_0(x)) - 1| + ((L_m e_0)(x) + 1)\omega(f; \delta_{m,x}) \leq M u_m(K) + (2 + u_m(K))\omega(f; \delta_m),$$

where

$$\delta_{m,x} = \sqrt{(L_m e_0)(x)[(L_m\psi_x^2)(x) + 2\alpha_m(L_m e_1)(x) + (\alpha_m^2 + 2x\alpha_m)(L_m e_0)(x)]},$$
$$\delta_m = \sqrt{(1 + u_m(K))[w_m(K) + 2\alpha_m(b + v_m(K) + (\alpha_m^2 + 2b\alpha_m)(1 + u_m(K))]}$$

and

$$M = \sup\{|f(x)| : x \in K\}.$$

Proof. We apply the Shisha-Mond Theorem (see [22] or [24]) for the operator $\mathcal{K}_m$ and taking the inequality from the proof of the Theorem 3.1 into account verified by $(\mathcal{K}_m\psi_x^2)(x)$ and Remark 2, the inequality (3.2) follows. □

Corollary 3.3. *If*

$$(3.3)\quad \sum_{k=0}^{p_m} \varphi_{m,k}(x) = 1$$

for any $x \in J$, then for any $f \in E(I \cap J) \cap C(I \cap J)$, any $x \in K = [a, b] \subset I \cap J$ and any $m \in \mathbb{N}$ we have that

$$(3.4)\quad |(\mathcal{K}_m f)(x) - f(x)| \leq 2\omega(f; \delta_{m,x}) \leq 2\omega(f; \delta'_m)$$

where $\delta'_m = \sqrt{w_m(K) + 2\alpha_m v_m(K) + \alpha_m^2 + 4b\alpha_m}$.

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Proof. It results from Theorem 3.2, because $(L_m e_0)(x) = 1$, for any $m \in \mathbb{N}$ and $x \in J$, so $u_m(K) = 0$, for any $m \in \mathbb{N}$. □

Remark 4. From the conditions of Theorem 3.2 we have that

$$|(\mathcal{K}_m f)(x) - f(x)| \le M u_m(K) + (2 + u_m(K))\omega(f; \delta_m)$$

and because $\lim_{m\to\infty} \delta_m = 0$, it results that the convergence $\lim_{m\to\infty} (K_m f)(x) = f(x)$ is uniform on K.

In the following, by particularisation of the sequence $y_{m,k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, p_m\} \cap \mathbb{N}_0$ and applying Theorem 3.1 and Corollary 3.3, we can obtain a convergence and approximation theorem for the new operators. In Applications 1 – 2, let $p_m = m$, $\varphi_{m,k}(x) = p_{m,k}(x)$, where $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$ and $K = [0, 1]$.

Application 1. *If $I = J = [0, 1]$, $E(I) = F(J) = C([0, 1])$, $x_{m,k} = \frac{k}{m}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$, we obtain the Bernstein operators. We have that $u_m([0, 1]) = 0$, $v_m([0, 1]) = 0$ and $w_m([0, 1]) = \frac{1}{4m}$, $m \in \mathbb{N}$. We consider the nodes $y_{m,k} = \frac{\sqrt{k(k+1)}}{m}$, $m \in \mathbb{N}$, $k \in \{0, 1, \dots, m\}$. Then it is verified immediately that $\alpha_m = \frac{1}{m+\sqrt{m(m+1)}}$, $m \in \mathbb{N}$ and $\lim_{m\to\infty} \alpha_m = 0$. In this case, the operators $(\mathcal{K}_m)_{m\ge1}$ have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{\sqrt{k(k+1)}}{m}\right),$$

$f \in C([0, 1])$, $x \in [0, 1]$, $m \in \mathbb{N}$ and $\delta'_m < \sqrt{\frac{5}{4m} + \frac{2}{m+\sqrt{m(m+1)}}} < \frac{3}{2\sqrt{m}}$, $m \in \mathbb{N}$.

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Application 2. *We study a particular case of the Stancu operators. Let* $\alpha = 10$ *and* $\beta = -\frac{1}{2}$. *We obtain* $I = [0, 22]$ *and for any* $f \in C([0, 22])$, $x \in [0, 1]$ *and* $m \in \mathbb{N}$

$$\left(P_m^{(10,-1/2)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{2k + 20}{2m - 1}\right).$$

We consider the nodes $y_{m,k} = \frac{(4k+40)m}{(2m-1)^2}$. *In this case, the operators* $(\mathcal{K}_m)_{m \geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{m(4k + 40)}{(2m - 1)^2}\right),$$

where $f \in C([0, 22])$, $x \in [0, 1]$, $m \in \mathbb{N}$ *and* $\delta'_m < \frac{\sqrt{36m^3 + 2220m^2 - 399m + 81}}{(2m-1)^2} < \frac{45}{\sqrt{2m-1}}$, $m \in \mathbb{N}$.

Application 3. *If* $I = J = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(J) = C([0, \infty))$, $K = [0, b]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = e^{-mx} \frac{(mx)^k}{k!}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *we obtain the Mirakjan-Favard-Szász operators and we have that* $u_m(K) = 0$, $v_m(K) = 0$ *and* $w_m(K) = \frac{b}{m}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{2k(k+1)}{m(2k+1)}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{2m}$, $m \in \mathbb{N}$. *In this case, the operators* $(\mathcal{K}_m)_{m \geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = e^{-mx} \sum_{k=0}^{\infty} \frac{(mx)^k}{k!} f\left(\frac{2k(k + 1)}{m(2k + 1)}\right),$$

where $f \in C_2([0, \infty))$, $x \in [0, \infty)$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{3b}{m} + \frac{1}{4m^2}}$, $m \in \mathbb{N}$.

Application 4. *Let* $I = J = [0, \infty)$, $E(I) = C_2([0, \infty))$, $F(J) = C([0, \infty))$, $K = [0, b]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$, $\varphi_{m,k}(x) = (1 + x)^{-m} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k$, $m \in \mathbb{N}$, $k \in$

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$\mathbb{N}_0$. *In this case, we obtain the Baskakov operators and we have that* $u_m(K) = 0$, $v_m(K) = 0$ *and* $w_m(K) = \frac{b(1+b)}{2m}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\sqrt{4k^2+4k+2}}{2m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{m\sqrt{2}}$. *The operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{\sqrt{4k^2+4k+2}}{2m}\right),$$

where $f \in C_2([0,\infty))$, $x \in [0,\infty)$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{b(b+1+2\sqrt{2})}{m} + \frac{1}{2m^2}}$, $m \in \mathbb{N}$.

Application 5. *If* $I = J = [0,\infty)$, $E(I) = F(J) = C([0,\infty))$, $K = [0,b]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m}$,

$$\varphi_{m,k}(x) = \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{\frac{-(k+m)x}{1+x}}, \qquad m \in \mathbb{N}, k \in \mathbb{N}_0,$$

we obtain the Ismail-May operators and we have that $u_m(K) = 0$, $v_m(K) = 0$ *and* $w_m(K) = \frac{b(1+b)^2}{m}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\sqrt[3]{k^2(k+1)}}{m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$ *and we have that* $\alpha_m = \frac{1}{3m}$. *In this case, the operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = e^{\frac{-mx}{1+x}} \sum_{k=0}^{\infty} \frac{m(m+k)^{k-1}}{k!} \left(\frac{x}{1+x}\right)^k e^{-\frac{kx}{1+x}} f\left(\frac{\sqrt[3]{k^2(k+1)}}{m}\right),$$

where $f \in C([0,\infty))$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{b(7+6b+3b^2)}{3m} + \frac{1}{9m^2}}$, $m \in \mathbb{N}$.

Application 6. *We consider* $I = J = [0,\infty)$, $E(I) = F(J) = C_B([0,\infty))$, $K = [0,b]$, $p_m = m$, $x_{m,k} = \frac{k}{m+1-k}$, $\varphi_{m,k}(x) = \frac{1}{(1+x)^m}\binom{m}{k}x^k$, $m \in \mathbb{N}$, $k \in \{0,1,\ldots,m\}$. *In this case we obtain the Bleimann-Butzer-Hahn operators and we*

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have that $u_m(K) = 0$, $v_m(K) = b\left(\frac{b}{1+b}\right)^m$ *and* $w_m(K) = \frac{4b(1+b)^2}{m+2}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{\beta_m k}{m+1-k}$, $m \in \mathbb{N}$, $k \in \{0, 1, \ldots, m\}$, *where* $(\beta_m)_{m\geq 1}$ *is a sequence of positive real numbers such that* $\lim_{m\to\infty} m(1-\beta_m) = 0$ *and we have* $\alpha_m = m|1-\beta_m|$, $m \in \mathbb{N}$. *The operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = (1+x)^{-m} \sum_{k=0}^{m} \binom{m}{k} x^k f\left(\frac{\beta_m k}{m+1-k}\right),$$

where $x \in [0,\infty)$, $m \in \mathbb{N}$, $f \in C_B([0,\infty))$.

Application 7. *If* $I = J = [0,1]$, $E(I) = B([0,1])$, $F(J) = C([0,1])$, $K = [0,1]$, $p_m = \infty$, $x_{m,k} = \frac{k}{m+k}$, $\varphi_{m,k}(x) = \binom{m+k}{k}(1-x)^{m+1}x^k$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *we obtain the Meyer-König and Zeller operators and we have that* $u_m([0,1]) = 0$, $v_m([0,1]) = 0$ *and* $w_m([0,1]) = \frac{1}{4(m+1)}$, $m \in \mathbb{N}$. *We consider the nodes* $y_{m,k} = \frac{k+\beta_m}{m+k+\beta_m}$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, *where* $(\beta_m)_{m\geq 1}$ *is a sequence of positive real numbers so that* $\lim_{m\to\infty} \frac{\beta_m}{m+\beta_m} = 0$. *Then it is verified immediately that* $\alpha_m = \frac{\beta_m}{m+\beta_m}$, $m \in \mathbb{N}$ *and the operators* $(\mathcal{K}_m)_{m\geq 1}$ *have the form*

$$(\mathcal{K}_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k}{k} (1-x)^{m+1} x^k f\left(\frac{k+\beta_m}{m+k+\beta_m}\right),$$

where $f \in B([0,1])$, $x \in [0,1]$, $m \in \mathbb{N}$ *and* $\delta'_m = \sqrt{\frac{1}{4(m+1)} + \frac{\beta_m(4m+5\beta_m)}{(m+\beta_m)^2}}$, $m \in \mathbb{N}$.

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