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NOTE ON AN OPEN PROBLEM OF FENG QI

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ABSTRACT. In this paper, an integral inequality is studied. An answer to an open problem proposed by Feng Qi is given.

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In [5], Qi studied a very interesting integral inequality and proved the following result

Theorem 1. *Let $f(x)$ be continuous on $[a, b]$, differentiable on (a, b) and $f(a) = 0$. If $f'(x) \geq 1$ for $x \in (a, b)$, then*

$$\int_a^b [f(x)]^3\,dx \geq \left[\int_a^b f(x)\,dx\right]^2. \tag{1}$$

If $0 \leq f'(x) \leq 1$, then the inequality (1) reverses.

Qi extended this result to a more general case [5], and obtained the following inequality (2).

Theorem 2. *Let n be a positive integer. Suppose $f(x)$ has continuous derivative of the n-th order on the interval $[a, b]$ such that $f^{(i)}(a) \geq 0$ where $0 \leq i \leq n-1$, and $f^{(n)}(x) \geq n!$, then*

$$\int_a^b [f(x)]^{n+2}\,dx \geq \left[\int_a^b f(x)\,dx\right]^{n+1}. \tag{2}$$

Qi then proposed an open problem: Under what condition is the inequality (2) still true if n is replaced by any positive real number r?

Some new results on this subject can be found in [1], [2], [3], and [4].

We now give an answer to Qi's open problem. The following result is a generalization of Theorem 1.

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Theorem 3. *Let p be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a)=0$. If $[f^{\frac{1}{p}}]'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then*

$$\int_a^b [f(x)]^{p+2}\,dx \geq \left[\int_a^b f(x)\,dx\right]^{p+1}. \tag{3}$$

If $0 \leq [f^{\frac{1}{p}}]'(x) \leq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then the inequality (3) reverses.

Proof. Suppose that $[f^{\frac{1}{p}}]'(x) \geq 0$, $x \in (a,b)$. Then $f^{\frac{1}{p}}(x)$ is a non-decreasing function. It follows that $f(x) \geq 0$ for all $x \in (a,b]$.

If $[f^{\frac{1}{p}}]'(x) \geq (p+1)^{\frac{1}{p}-1}$ for $x \in (a,b)$, then $f(x) > 0$ for $x \in (a,b]$. Thus both sides of (3) are not 0. Now consider the quotient of both sides of (3). By using Cauchy's Mean Value Theorem twice, we have

$$\frac{\int_a^b [f(x)]^{p+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{p+1}} = \frac{[f(b_1)]^{p+1}}{(p+1)\left[\int_a^{b_1} f(x)\,dx\right]^p} \quad (a < b_1 < b) \tag{4}$$

$$= \left(\frac{[f(b_1)]^{1+\frac{1}{p}}}{(p+1)^{\frac{1}{p}}\int_a^{b_1} f(x)\,dx}\right)^p \tag{5}$$

$$= \left(\frac{(1+\frac{1}{p})[f(b_2)]^{\frac{1}{p}} f'(b_2)}{(p+1)^{\frac{1}{p}} f(b_2)}\right)^p \quad (a < b_2 < b_1) \tag{6}$$

$$= \left((1+p)^{1-\frac{1}{p}}[f^{\frac{1}{p}}]'(b_2)\right)^p. \tag{7}$$

$$\geq 1. \tag{8}$$

So the inequality (3) holds.

If $f \equiv 0$ on $[a,b]$, then it is trivial that the equation in (3) holds. Suppose now that f is not identically 0 on $[a,b]$. Since $f(x)$ is non-decreasing and non-negative, we may assume $f(x) > 0$, $x \in (a,b]$ (otherwise we can find a_1 such that $a_1 < b$, $f(a_1) = 0$ and $f(x) > 0$ for $a_1 < x < b$ and hence we only need to consider f on $(a_1, b]$). This implies that both sides of (3) are not 0. Now if $0 \leq [f^{\frac{1}{p}}]'(x) \leq (p+1)^{\frac{1}{p}-1}$, then $(1+p)^{1-\frac{1}{p}}[f^{\frac{1}{p}}]'(b_2) \leq 1$, which, together with (7), implies that the inequality (3) reverses. □

Note that if $p = 1$, then (3) becomes (1). So Theorem 1 is just a special case of Theorem 3.

In Theorem 1, we see that if $f'(x) = 1$, then $f(x) = x - a$ and the equation in (1) holds. A very natural question can be asked the same way: For what polynomial $f(x) = C(x-a)^n$ does the equation in (2) hold? It is easy to see that $C = \frac{1}{(n+1)^{(n-1)}}$. The n-th derivative of this polynomial is a constant $\frac{n!}{(n+1)^{(n-1)}}$. This motivates the following theorem.

Theorem 4. *Suppose $f(x)$ has derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\ldots,n-1$. If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is increasing, then the inequality (2) holds. If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is decreasing, then the inequality (2) reverses.*

Proof. Suppose that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$. It is easy to see that

$$f(x) \geq \frac{(x-a)^n}{(n+1)^{n-1}}.$$

Using the same argument as in the proof of Theorem 3, we have

$$\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^n} \quad (a < b_1 < b) \tag{9}$$

$$\geq \frac{\frac{(b_1-a)^n}{(n+1)^{n-1}}[f(b_1)]^n}{(n+1)\left[\int_a^{b_1} f(x)\,dx\right]^n} \tag{10}$$

$$= \left(\frac{(b_1-a)f(b_1)}{(n+1)\int_a^{b_1} f(x)\,dx}\right)^n. \tag{11}$$

Now for the term in (11), by using Cauchy's Mean Value Theorem several times, we will have

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,dx} = 1 + \frac{(b_2-a)f'(b_2)}{f(b_2)} \quad (a < b_2 < b_1) \tag{12}$$

$$= 2 + \frac{(b_3-a)f''(b_3)}{f'(b_3)} \quad (a < b_3 < b_2) \tag{13}$$

$$\dots \tag{14}$$

$$= n + \frac{(b_{n+1}-a)f^{(n)}(b_{n+1})}{f^{(n-1)}(b_{n+1})} \quad (a < b_{n+1} < b_n). \tag{15}$$

But

$$f^{(n-1)}(t) = f^{(n-1)}(t) - f^{(n-1)}(a) = f^{(n)}(t_1)(t-a)$$

for some $t_1 \in (a,t)$. If $f^{(n)}(x)$ is increasing, then $f^{(n)}(t_1) \leq f^{(n)}(t)$. Therefore

$$f^{(n-1)}(t) \leq f^{(n)}(t)(t-a). \tag{16}$$

Applying (16) to (15) yields

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,dx} \geq n+1. \tag{17}$$

(2) follows from (17) and (11).

Suppose that $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ and $f^n(x)$ is decreasing. It is clear $f^{(n-1)}(t)$ is increasing. If $f^{(n-1)}(t) = 0$ for some $t \in (a,b)$, then $f^{(n-1)}(s) = 0$ for $s \in (a,t)$. Hence $f^{(i)}(s) = 0$ for $s \in (a,t)$ and $0 \leq s \leq n-1$. So we can assume that $f^{(n-1)}(x) \neq 0$ for $x \in (a,b)$. By Rolle's Theorem, this means that $f^{(i)}(x) \neq 0$ for $x \in (a,b)$ and for $0 \leq i \leq n-1$. Now that the inequalities (10) and (16) reverse, it follows that the inequality (17) reverses, so does (2). $\square$

Unfortunately there is an additional hypothesis on monotonicity in Theorem 4. Our conjecture is that this hypothesis could be dropped. But we are not able to prove it for the moment. However, we have

Theorem 5. *Suppose $f(x)$ has derivative of the n-th order on the interval $[a,b]$ such that, $f^{(i)}(a) = 0$ for $i = 0,1,2,\dots,n-1$. If $f^{(n)}(x) \geq \frac{(n+1)!}{n^n}$, then the inequality (2) holds.*

Proof. If $f(x) \geq \frac{(n+1)!}{n^n}$, then

$$f(x) \geq \frac{n+1}{n^n}(x-a)^n. \tag{18}$$

(11) now becomes

$$\frac{\int_a^b [f(x)]^{n+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{n+1}} \geq \left(\frac{(b_1-a)f(b_1)}{n\int_a^{b_1} f(x)\,dx}\right)^n. \tag{19}$$

Note that all the terms in (15) are positive, so we have

$$\frac{(b_1-a)f(b_1)}{\int_a^{b_1} f(x)\,dx} \geq n. \tag{20}$$

The inequality (2) follows from (19) and (20). □

The same argument can be used to prove the following result obtained by Pečarić and Pejković [3, Theorem 2].

Theorem 6. *Let p be a positive number and $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f(a) \geq 0$. If $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$, then the inequality (3) holds.*

Proof. Suppose that $f'(x) \geq p(x-a)^{p-1}$ for $x \in (a,b)$. Consider the quotient of the two sides of (3). By using Cauchy's Mean Value Theorem three times, we have

$$\frac{\int_a^b [f(x)]^{p+2}\,dx}{\left[\int_a^b f(x)\,dx\right]^{p+1}} = \frac{[f(b_1)]^{p+1}}{(p+1)\left[\int_a^{b_1} f(x)\,dx\right]^p} \quad (a < b_1 < b) \tag{21}$$

$$= \frac{[f(b_2)]^p f'(b_2)}{(p-1)\left[\int_a^{b_2} f(x)\,dx\right]^{p-1}} \quad (a < b_2 < b_1) \tag{22}$$

$$\geq \left(\frac{f(b_2)(b_2-a)}{\int_a^{b_2} f(x)\,dx)}\right)^{p-1} \tag{23}$$

$$= \left(1 + \frac{f'(b_3)(b_3-a)}{f(b_3)}\right)^{p-1} \tag{24}$$

$$\geq 1. \tag{25}$$

This completes the proof. □

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