

AN INTEGRAL INEQUALITY FOR 3-CONVEX FUNCTIONS

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Abstract: In this paper, an integral inequality and an application of it, that imply the Chebyshev functional for two 3-convex (3-concave) functions, are given.



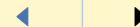
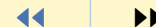
**Integral Inequality for
3-Convex Functions**

Vlad Ciobotariu-Boer

vol. 9, iss. 4, art. 98, 2008

[Title Page](#)

[Contents](#)



Page 1 of 21

[Go Back](#)

[Full Screen](#)

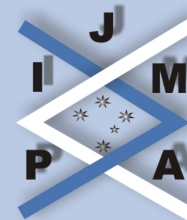
[Close](#)

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Contents

1	Introduction	3
2	Results	4
3	An Application	15



**Integral Inequality for
3-Convex Functions**

Vlad Ciobotariu-Boer

vol. 9, iss. 4, art. 98, 2008

Title Page

Contents



Page 2 of 21

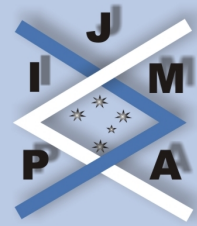
Go Back

Full Screen

Close

journal of **inequalities**
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Title Page

Contents



Page 3 of 21

Go Back

Full Screen

Close

1. Introduction

For two Lebesgue functions $f, g : [a, b] \rightarrow \mathbb{R}$, consider the Chebyshev functional

$$C(f, g) := \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \cdot \int_a^b g(x)dx.$$

In 1972, A. Lupuş [2] showed that if f, g are convex functions on the interval $[a, b]$, then

$$(1.1) \quad C(f, g) \geq \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x)dx,$$

with equality when at least one of the functions f, g is a linear function on $[a, b]$. He proved this result using the following lemma:

Lemma 1.1. *If $f, g : [a, b] \rightarrow \mathbb{R}$ are convex functions on the interval $[a, b]$, then*

$$(1.2) \quad [F(e^2) - F(e)^2]F(fg) - F(e^2)F(f)F(g) \\ \geq F(ef)F(eg) - F(e)[F(f)F(eg) + F(ef)F(g)],$$

where F is an isotonic positive linear functional, defined by one of the following relations:

$$(1.3) \quad F(f) := \frac{1}{b-a} \int_a^b f(x)dx, \quad F(f) := \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}, \quad F(f) := \sum_{i=1}^n p_i f(x_i)$$

($x_i \in [a, b]; p_i \geq 0, i = 1, 2, \dots, n, \sum_{i=1}^n p_i = 1$), $p : [a, b] \rightarrow \mathbb{R}$ is a positive, integrable function on $[a, b]$ and $e(x) = x, x \in [a, b]$. If f or g is a linear function, then the equality holds in (1.2).

In this note, we provide a lower bound for the Chebyshev functional in the case of two 3-convex (3-concave) functions f and g .



Title Page

Contents



Page 4 of 21

Go Back

Full Screen

Close

2. Results

Note that the inequality (1.2) can be written in the form:

$$(2.1) \quad \begin{vmatrix} 1 & F(e) & F(g) \\ F(e) & F(e^2) & F(eg) \\ F(f) & F(ef) & F(fg) \end{vmatrix} \geq 0.$$

The following lemma holds.

Lemma 2.1. *If $f, g : [a, b] \rightarrow \mathbb{R}$ are 3-convex (3-concave) functions on the interval $[a, b]$, then*

$$(2.2) \quad \begin{vmatrix} 1 & F(e) & F(e^2) & F(g) \\ F(e) & F(e^2) & F(e^3) & F(eg) \\ F(e^2) & F(e^3) & F(e^4) & F(e^2g) \\ F(f) & F(ef) & F(e^2f) & F(fg) \end{vmatrix} \geq 0,$$

where $e^i(x) = x^i$, $x \in [a, b]$, $i = \overline{1, 4}$ and F is defined by (1.3).

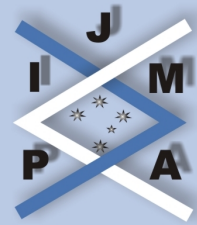
If f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.2) holds.

If f or g is a polynomial function of degree at most two, then the equality holds in (2.2).

Proof. Let $[x, y, z, t; f]$ be the divided difference of a certain function f . If f and g are 3-convex (3-concave) on the interval $[a, b]$, then we have

$$(2.3) \quad [x, y, z, t; f] \cdot [x, y, z, t; g] \geq 0,$$

for all distinct points x, y, z, t from $[a, b]$.



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 5 of 21

Go Back

Full Screen

Close

When f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.3) holds.

In the following we prove (2.2) in the case when both functions f and g are 3-convex (3-concave). The inequality (2.3) is equivalent to

$$(2.4) \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ f(x) & f(y) & f(z) & f(t) \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & t \\ x^2 & y^2 & z^2 & t^2 \\ g(x) & g(y) & g(z) & g(t) \end{vmatrix} \geq 0,$$

with true equality holding when at least one of f and g is a polynomial function of degree at most two.

Note that the function F defined by (1.3) has the property $F(1) = 1$. In order to put in evidence the variable u , we write F_u instead of F .

Now, using the fact that F is a linear positive functional, by applying successively on (2.4) the functionals F_x, F_y, F_z and then F_t , we obtain the inequality (2.2). For instance, if

$$A = A(x, y, z, t, f, g) := \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix}^2 \cdot f(x)g(x),$$

then

$$F_x(A) = \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix}^2 \cdot F(fg),$$

$$F_t F_z F_y F_x(A) = 6 \cdot \begin{vmatrix} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{vmatrix} \cdot F(fg)$$

and if

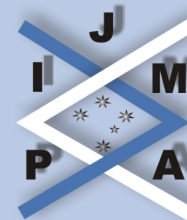
$$B = B(x, y, z, t, f, g) := \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ x & z & t \\ x^2 & z^2 & t^2 \end{vmatrix} \cdot f(x)g(y),$$

then

$$F_x(B) = \begin{vmatrix} 1 & 1 & 1 \\ y & z & t \\ y^2 & z^2 & t^2 \end{vmatrix} \cdot g(y) \cdot \begin{vmatrix} F(f) & 1 & 1 \\ F(ef) & z & t \\ F(e^2f) & z^2 & t^2 \end{vmatrix},$$

$$\begin{aligned} F_t F_z F_y F_x(B) &= 2 \cdot \begin{vmatrix} 1 & F(e) & F(g) \\ F(e) & F(e^2) & F(eg) \\ F(e^2) & F(e^3) & F(e^2g) \end{vmatrix} \cdot F(e^2f) \\ &+ 2 \cdot \begin{vmatrix} 1 & F(g) & F(e^2) \\ F(e) & F(eg) & F(e^3) \\ F(e^2) & F(e^2g) & F(e^4) \end{vmatrix} \cdot F(ef) \\ &+ 2 \cdot \begin{vmatrix} F(g) & F(e) & F(e^2) \\ F(eg) & F(e^2) & F(e^3) \\ F(e^2g) & F(e^3) & F(e^4) \end{vmatrix} \cdot F(f). \end{aligned}$$

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[Title Page](#)

[Contents](#)

◀◀ ▶▶

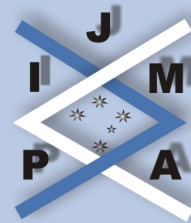
◀ ▶

Page 6 of 21

[Go Back](#)

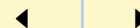
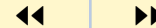
[Full Screen](#)

[Close](#)



Title Page

Contents



Page 7 of 21

Go Back

Full Screen

Close

Theorem 2.2. If f, g are 3-convex (3-concave) functions on the interval $[a, b]$, then

$$(2.5) \quad C(f, g) \geq \frac{180}{(b-a)^6} \int_a^b q(x)f(x)dx \cdot \int_a^b q(x)g(x)dx \\ + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x)dx,$$

where

$$q(x) = \left(x - \frac{a+b}{2} - \frac{b-a}{2\sqrt{3}}\right) \left(x - \frac{a+b}{2} + \frac{b-a}{2\sqrt{3}}\right).$$

If f is 3-convex (3-concave) and g is 3-concave (3-convex) then the reverse of the inequality in (2.5) holds.

The equality in (2.5) holds when at least one of f or g is a polynomial function of degree at most two on $[a, b]$.

Proof. We choose

$$(2.6) \quad F(f) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Then

$$(2.7) \quad F(e) = \frac{a+b}{2}, \quad F(e^2) = \frac{a^2 + ab + b^2}{3},$$

$$F(e^3) = \frac{a^3 + a^2b + ab^2 + b^3}{4}, \quad F(e^4) = \frac{a^4 + a^3b + a^2b^2 + ab^3 + b^4}{5}.$$



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 8 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

Note that the inequality (2.2) can be written as

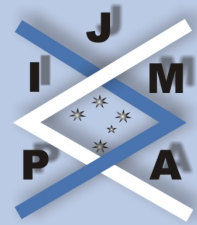
$$\begin{aligned}
 (2.8) \quad & \left| \begin{array}{ccc} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{array} \right| \cdot F(fg) - \left| \begin{array}{cc} 1 & F(e) \\ F(e) & F(e^2) \end{array} \right| \cdot F(e^2f)F(e^2g) \\
 & - \left| \begin{array}{cc} 1 & F(e^2) \\ F(e^2) & F(e^4) \end{array} \right| \cdot F(ef)F(eg) - \left| \begin{array}{cc} F(e^2) & F(e^3) \\ F(e^3) & F(e^4) \end{array} \right| \cdot F(f)F(g) \\
 & + \left| \begin{array}{cc} 1 & F(e) \\ F(e^2) & F(e^3) \end{array} \right| \cdot [F(e^2f)F(eg) + F(ef)F(e^2g)] \\
 & - \left| \begin{array}{cc} F(e) & F(e^2) \\ F(e^2) & F(e^3) \end{array} \right| \cdot [F(e^2f)F(g) + F(f)F(e^2g)] \\
 & + \left| \begin{array}{cc} F(e) & F(e^2) \\ F(e^3) & F(e^4) \end{array} \right| \cdot [F(ef)F(g) + F(f)F(eg)] \geq 0.
 \end{aligned}$$

By calculation, we find

$$(2.9) \quad \left| \begin{array}{ccc} 1 & F(e) & F(e^2) \\ F(e) & F(e^2) & F(e^3) \\ F(e^2) & F(e^3) & F(e^4) \end{array} \right| = \frac{(b-a)^6}{2160},$$

$$(2.10) \quad \left| \begin{array}{cc} 1 & F(e^2) \\ F(e^2) & F(e^4) \end{array} \right| = \frac{(b-a)^2(4a^2 + 7ab + 4b^2)}{45},$$

$$(2.11) \quad \left| \begin{array}{cc} F(e^2) & F(e^3) \\ F(e^3) & F(e^4) \end{array} \right| = \frac{(b-a)^2(a^4 + 4a^3b + 10a^2b^2 + 4ab^3 + b^4)}{240},$$



Title Page

Contents



Page 9 of 21

Go Back

Full Screen

Close

$$(2.12) \quad \left| \begin{array}{cc} 1 & F(e) \\ F(e^2) & F(e^3) \end{array} \right| = \frac{(b-a)^2(a+b)}{12},$$

$$(2.13) \quad \left| \begin{array}{cc} F(e) & F(e^2) \\ F(e^2) & F(e^3) \end{array} \right| = \frac{(b-a)^2(a^2+4ab+b^2)}{72},$$

$$(2.14) \quad \left| \begin{array}{cc} F(e) & F(e^2) \\ F(e^3) & F(e^4) \end{array} \right| = \frac{(b-a)^2(a^3+4a^2b+4ab^2+b^3)}{60}.$$

The relations (2.7) – (2.14) give us

$$(2.15) \quad \begin{aligned} & \frac{(b-a)^5}{2160} \int_a^b f(x)g(x)dx \\ & - \frac{a^4+4a^3b+10a^2b^2+4ab^3+b^4}{240} \int_a^b f(x)dx \int_a^b g(x)dx \\ & \geq \frac{1}{12} \int_a^b x^2 f(x)dx \int_a^b x^2 g(x)dx \\ & + \frac{4a^2+7ab+4b^2}{45} \int_a^b x f(x)dx \int_a^b x g(x)dx \\ & - \frac{a+b}{12} \left[\int_a^b x^2 f(x)dx \int_a^b x g(x)dx + \int_a^b x f(x)dx \int_a^b x^2 g(x)dx \right] \\ & + \frac{a^2+4ab+b^2}{72} \left[\int_a^b x^2 f(x)dx \int_a^b g(x)dx \right. \\ & \quad \left. + \int_a^b f(x)dx \int_a^b x^2 g(x)dx \right] \end{aligned}$$

$$- \frac{a^3 + 4a^2b + 4ab^2 + b^3}{60} \left[\int_a^b xf(x)dx \int_a^b g(x)dx + \int_a^b f(x)dx \int_a^b xg(x)dx \right],$$

or

$$(2.16) \quad C(f, g) \geq \frac{180}{(b-a)^6} \left\{ \int_a^b x^2 f(x)dx \int_a^b x^2 g(x)dx + \frac{4(4a^2 + 7ab + 4b^2)}{15} \int_a^b xf(x)dx \int_a^b xg(x)dx + \frac{2a^4 + 10a^3b + 21a^2b^2 + 10ab^3 + 2b^4}{540} \cdot \int_a^b f(x)dx \int_a^b g(x)dx - \frac{a+b}{12} \left[\int_a^b x^2 f(x)dx \int_a^b xg(x)dx + \int_a^b xf(x)dx \int_a^b x^2 g(x)dx \right] + \frac{a^2 + 4ab + b^2}{72} \left[\int_a^b x^2 f(x)dx \int_a^b g(x)dx + \int_a^b f(x)dx \int_a^b x^2 g(x)dx \right] - \frac{a^3 + 4a^2b + 4ab^2 + b^3}{60} \left[\int_a^b xf(x)dx \int_a^b g(x)dx + \int_a^b f(x)dx \int_a^b xg(x)dx \right] \right\}.$$



Title Page

Contents

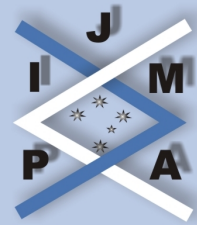


Page 10 of 21

Go Back

Full Screen

Close



Title Page

Contents



Page 11 of 21

Go Back

Full Screen

Close

The last inequality can be written as

$$\begin{aligned}
 (2.17) \quad C(f, g) &\geq \frac{180}{(b-a)^6} \int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 g(x) dx \\
 &\quad - \frac{15}{(b-a)^4} \cdot \left[\int_a^b \left(x - \frac{a+b}{2}\right)^2 f(x) dx \cdot \int_a^b g(x) dx + \right. \\
 &\quad \left. + \int_a^b f(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right)^2 g(x) dx \right] \\
 &\quad + \frac{5}{4(b-a)^2} \int_a^b f(x) dx \cdot \int_a^b g(x) dx \\
 &\quad + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx,
 \end{aligned}$$

which is equivalent to (2.5). □

Corollary 2.3. *Let f and g be as in Theorem 2.2 and assume that*

$$(2.18) \quad f(x) = -f(a+b-x)$$

or

$$(2.19) \quad g(x) = -g(a+b-x)$$

for all x from $[a, b]$. Then Lupas' inequality holds.

Proof. Note that the function denoted by q in Theorem 2.2 is symmetric about $x = \frac{a+b}{2}$, namely

$$q(x) = q(a+b-x),$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 21

Go Back

Full Screen

Close

for all x from $[a, b]$.

Assume that (2.18) is satisfied. Then we have

$$\begin{aligned}
 (2.20) \quad \int_a^b q(x)f(x)dx &= \frac{1}{2} \int_a^b q(x)[f(x) - f(a+b-x)]dx \\
 &= \frac{1}{2} \int_a^b q(x)f(x)dx - \frac{1}{2} \int_a^b q(a+b-x)f(a+b-x)dx \\
 &= \frac{1}{2} \int_a^b q(x)f(x)dx - \frac{1}{2} \int_a^b q(t)f(t)dt = 0.
 \end{aligned}$$

From (2.5) and (2.20), we deduce (1.1). □

Note that the condition (2.3) is important. The same results are valid if we suppose that this (or its reverse) is satisfied. Thus, we obtain a more general result:

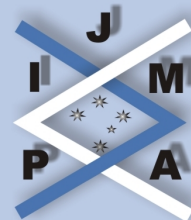
Theorem 2.4. *If the functions f and g are integrable on the interval $[a, b]$ and satisfy (2.3) (or its reverse), then we have (2.5) (or its reverse).*

The equality in (2.5) holds when at least one of f or g is a polynomial function of degree at most two on $[a, b]$.

Corollary 2.5. *If the function f is integrable on $[a, b]$, then we have*

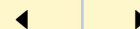
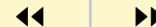
$$\begin{aligned}
 (2.21) \quad (b-a) \int_a^b f^2(x)dx - \left(\int_a^b f(x)dx \right)^2 \\
 \geq \frac{12}{(b-a)^2} \left(\int_a^b \left(x - \frac{a+b}{2} \right) f(x)dx \right)^2 + \frac{180}{(b-a)^4} \left(\int_a^b q(x)f(x)dx \right)^2,
 \end{aligned}$$

where $q(x)$ is defined in Theorem 2.2.



Title Page

Contents



Page 13 of 21

Go Back

Full Screen

Close

Proof. Considering $g(x) = f(x)$ in (2.5), we find the inequality (2.21). □

Remark 1. The inequality (2.21) is better than the well-known inequality

$$(2.22) \quad (b-a) \int_a^b f^2(x) dx \geq \left(\int_a^b f(x) dx \right)^2,$$

valid for all integrable functions f on $[a, b]$.

Corollary 2.6. *If the functions f, g satisfy the following conditions:*

(i) f, g are 3-convex (3-concave) functions on $[a, b]$;

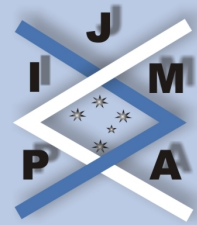
(ii) f, g are differentiable functions on $[a, b]$,

then we have

$$(2.23) \quad \int_a^b f'(x)g'(x)dx \geq \frac{[f(b) - f(a)][g(b) - g(a)]}{b-a} \\ + \frac{12}{b-a} \left(\frac{1}{b-a} \int_a^b f(x)dx - \frac{f(a) + f(b)}{2} \right) \\ \times \left(\frac{1}{b-a} \int_a^b g(x)dx - \frac{g(a) + g(b)}{2} \right).$$

Proof. In (1.1), we use the fact that if f, g are 3-convex functions on $[a, b]$, then f', g' are convex functions on $[a, b]$. We have

$$\frac{1}{b-a} \int_a^b f'(x)g'(x)dx - \frac{1}{(b-a)^2} \int_a^b f'(x)dx \cdot \int_a^b g'(x)dx \\ \geq \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2} \right) f'(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2} \right) g'(x)dx,$$



which is equivalent to (2.23). □

Remark 2. If, in addition, f and g are convex (concave) on $[a, b]$, then the inequality (2.23) is better than the inequality

$$(2.24) \quad \int_a^b f'(x) \cdot g'(x) dx \geq \frac{[f(b) - f(a)][g(b) - g(a)]}{b - a},$$

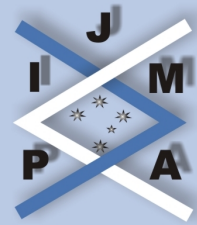
which is valid for all convex (concave) functions f, g on $[a, b]$.

Remark 3. Lemma 2.1 can be generalized for n -convex functions, obtaining a result similar to (2.2), from where the inequality

$$(2.25) \quad \left| \begin{array}{cccccc} b - a & \frac{b^2 - a^2}{2} & \dots & \frac{b^n - a^n}{n} & \int_a^b g(x) dx \\ \frac{b^2 - a^2}{2} & \frac{b^3 - a^3}{3} & \dots & \frac{b^{n+1} - a^{n+1}}{n+1} & \int_a^b xg(x) dx \\ \dots & \dots & \dots & \dots & \dots \\ \frac{b^n - a^n}{n} & \frac{b^{n+1} - a^{n+1}}{n+1} & \dots & \frac{b^{2n-1} - a^{2n-1}}{2n-1} & \int_a^b x^{n-1}g(x) dx \\ \int_a^b f(x) dx & \int_a^b xf(x) dx & \dots & \int_a^b x^{n-1}f(x) dx & \int_a^b f(x)g(x) dx \end{array} \right| \geq 0,$$

holds for all integer numbers $n \geq 3$.

Some similar results related to the Chebyshev functional are given in [1] – [6].



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 15 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

3. An Application

Let f, g be two 3-time differentiable functions defined on a nonempty interval $[a, b]$.

Denote

$$m_1 = \inf_{x \in [a, b]} f^{(3)}(x), \quad M_1 = \sup_{x \in [a, b]} f^{(3)}(x),$$

$$m_2 = \inf_{x \in [a, b]} g^{(3)}(x), \quad M_2 = \sup_{x \in [a, b]} g^{(3)}(x).$$

Considering the functions $F_1, G_1, F_2, G_2 : [a, b] \rightarrow \mathbb{R}$, defined by

$$F_1(x) = \frac{m_1 x^3}{6} - f(x), \quad G_1(x) = \frac{m_2 x^3}{6} - g(x),$$

$$F_2(x) = \frac{M_1 x^3}{6} - f(x), \quad G_2(x) = \frac{M_2 x^3}{6} - g(x),$$

we note that these are 3-differentiable on $[a, b]$ and $F_1^{(3)}(x) \leq 0$, $G_1^{(3)}(x) \leq 0$, $F_2^{(3)}(x) \geq 0$, $G_2^{(3)}(x) \geq 0$ for all $x \in [a, b]$. Therefore F_1, G_1 are 3-concave on $[a, b]$ and F_2, G_2 are 3-convex on $[a, b]$.

Applying Theorem 2.2 we shall prove the following result:

Theorem 3.1. *Let f, g be two 3-differentiable functions on the nonempty interval $[a, b]$. Then, we have*

$$(3.1) \quad \left| L(f, g) - \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2} \right) r(x) h(x) dx \right. \\ \left. + \frac{(m_1 + M_1)(m_2 + M_2)}{403200} \cdot (b-a)^6 \right| \\ \leq \frac{(M_1 - m_1)(M_2 - m_2)}{403200} \cdot (b-a)^6,$$



where

$$(3.2) \quad L(f, g) = C(f, g)$$

$$\begin{aligned} & - \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\ & - \frac{180}{(b-a)^6} \int_a^b q(x) f(x) dx \cdot \int_a^b q(x) g(x) dx, \end{aligned}$$

$$(3.3) \quad h(x) = \frac{m_1 + M_1}{2} \cdot g(x) + \frac{m_2 + M_2}{2} \cdot f(x),$$

$$(3.4) \quad r(x) = \left(x - \frac{a+b}{2} - \frac{(b-a)\sqrt{15}}{10}\right) \left(x - \frac{a+b}{2} + \frac{(b-a)\sqrt{15}}{10}\right).$$

Proof. Applying Theorem 2.2, we have

$$(3.5) \quad \begin{aligned} C(F_1, G_1) & \geq \frac{180}{(b-a)^6} \int_a^b q(x) F_1(x) dx \cdot \int_a^b q(x) G_1(x) dx \\ & + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_1(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_1(x) dx, \end{aligned}$$

$$(3.6) \quad \begin{aligned} C(F_2, G_2) & \geq \frac{180}{(b-a)^6} \int_a^b q(x) F_2(x) dx \cdot \int_a^b q(x) G_2(x) dx \\ & + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_2(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_2(x) dx, \end{aligned}$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 16 of 21

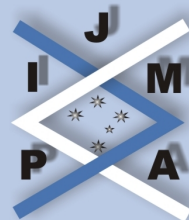
Go Back

Full Screen

Close

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Title Page

Contents



Page 17 of 21

Go Back

Full Screen

Close

$$(3.7) \quad C(F_1, G_2) \leq \frac{180}{(b-a)^6} \int_a^b q(x)F_1(x)dx \cdot \int_a^b q(x)G_2(x)dx \\ + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_1(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_2(x)dx,$$

$$(3.8) \quad C(F_2, G_1) \leq \frac{180}{(b-a)^6} \int_a^b q(x)F_2(x)dx \cdot \int_a^b q(x)G_1(x)dx \\ + \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_2(x)dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_1(x)dx,$$

where $q(x)$ is defined in Theorem 2.2.

By calculation, we find

$$(3.9) \quad C(F_1, G_1) = C(f, g) + \frac{m_1 m_2}{4032} (b-a)^2 (9a^4 + 20a^3b + 26a^2b^2 + 20ab^3 + 9b^4) \\ - \frac{1}{6(b-a)} \int_a^b x^3 [m_1 g(x) + m_2 f(x)] dx \\ + \frac{a^3 + a^2b + ab^2 + b^3}{24(b-a)} \int_a^b [m_1 g(x) + m_2 f(x)] dx,$$

$$(3.10) \quad \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) F_1(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) G_1(x) dx \\ = \frac{12}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2}\right) f(x) dx \cdot \int_a^b \left(x - \frac{a+b}{2}\right) g(x) dx \\ + \frac{m_1 m_2}{4800} (b-a)^2 (3a^2 + 4ab + 3b^2)^2$$



Title Page

Contents



Page 18 of 21

Go Back

Full Screen

Close

$$- \frac{3a^2 + 4ab + 3b^2}{20(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) [m_1g(x) + m_2f(x)]dx,$$

$$(3.11) \quad \frac{180}{(b-a)^6} \int_a^b q(x)F_1(x)dx \cdot \int_a^b q(x)G_1(x)dx \\ = \frac{180}{(b-a)^6} \int_a^b q(x)f(x)dx \cdot \int_a^b q(x)g(x)dx + \frac{m_1m_2}{2880}(b-a)^4(a+b)^2 \\ - \frac{a+b}{4(b-a)} \cdot \int_a^b q(x)[m_1g(x) + m_2f(x)]dx.$$

From (3.5) and (3.9) – (3.11), we obtain

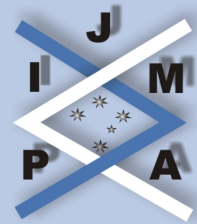
$$(3.12) \quad L(f, g) + \frac{m_1m_2}{100800}(b-a)^6 \\ \geq \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x)[m_1g(x) + m_2f(x)]dx.$$

In a similar way we can prove that the inequality (3.6) is equivalent to

$$(3.13) \quad L(f, g) + \frac{M_1M_2}{100800}(b-a)^6 \\ \geq \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x)[M_1g(x) + M_2f(x)]dx,$$

the inequality (3.7) is equivalent to

$$(3.14) \quad L(f, g) + \frac{m_1M_2}{100800}(b-a)^6$$



[Title Page](#)

[Contents](#)



Page 19 of 21

[Go Back](#)

[Full Screen](#)

[Close](#)

$$\leq \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x) [m_1 g(x) + M_2 f(x)] dx,$$

and the inequality (3.8) is equivalent to

$$(3.15) \quad L(f, g) + \frac{M_1 m_2}{100800} (b-a)^6 \\ \leq \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x) [M_1 g(x) + m_2 f(x)] dx.$$

From (3.12) and (3.13) we deduce

$$(3.16) \quad L(f, g) - \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x) h(x) dx \\ \geq -\frac{m_1 m_2 + M_1 M_2}{201600} (b-a)^6.$$

From (3.14) and (3.15) we find

$$(3.17) \quad L(f, g) - \frac{1}{6(b-a)} \int_a^b \left(x - \frac{a+b}{2}\right) r(x) h(x) dx \\ \leq -\frac{m_1 M_2 + M_1 m_2}{201600} (b-a)^6.$$

The inequalities (3.16) and (3.17) prove (3.1). \square

Corollary 3.2. *If f, g are 3-time differentiable on $[a, b]$ and symmetric about $x =$*



Title Page

Contents



Page 20 of 21

Go Back

Full Screen

Close

$\frac{a+b}{2}$, then we have

$$(3.18) \quad \left| L(f, g) + \frac{(m_1 + M_1)(m_2 + M_2)}{403200} (b - a)^6 \right| \leq \frac{(M_1 - m_1)(M_2 - m_2)}{403200} (b - a)^6.$$

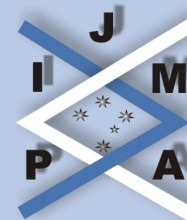
Proof. Note that the functions h and r defined on $[a, b]$ by (3.3) and (3.4) are symmetric about $x = \frac{a+b}{2}$. Hence, their product $h \cdot r$ is symmetric about $x = \frac{a+b}{2}$ and

$$(3.19) \quad \int_a^b \left(x - \frac{a+b}{2} \right) r(x) h(x) dx = 0.$$

From (3.1) and (3.19), we obtain (3.18). □

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Title Page

Contents



Page 21 of 21

Go Back

Full Screen

Close

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