Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 4, Article 141, 2006

SIMULTANEOUS APPROXIMATION FOR THE PHILLIPS-BÉZIER OPERATORS

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Received 28 July, 2006; accepted 17 November, 2006
Communicated by F. Qi

ABSTRACT. We study the simultaneous approximation properties of the Bézier variant of the well known Phillips operators and estimate the rate of convergence of the Phillips-Bézier operators in simultaneous approximation, for functions of bounded variation.

Key words and phrases: Rate of convergence, Bounded variation, Total variation, Simultaneous approximation.

2000 *Mathematics Subject Classification.* 41A25, 41A30.

1. INTRODUCTION

For $\alpha \geq 1$, the Phillips-Bézier operator is defined by

$$P_{n,\alpha}(f,x) = n\sum_{k=1}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_0^{\infty} p_{n,k-1}(t)f(t)dt + Q_{n,0}^{(\alpha)}(x)\, f(0), \tag{1.1}$$

ISSN (electronic): 1443-5756

204-06

where $n\in\mathbb{N},\ x\in[0,\infty)$,

$$Q_{n,k}^{(\alpha)}(x)=\left[J_{n,k}(x)\right]^{\alpha}-\left[J_{n,k+1}(x)\right]^{\alpha},\qquad J_{n,k}(x)=\sum_{j=k}^{\infty}p_{n,j}(x)\qquad\text{and}$$

$$p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}.$$

For $\alpha=1$, the operator (1.1) reduces to the Phillips operator [1]. Some approximation properties of the Phillips operators were recently studied by Finta and Gupta [2]. The rates of convergence in ordinary and simultaneous approximations on functions of bounded variation for the Phillips operators were estimated in [3], [4] and [5]. In the present paper we extend the earlier study and here we investigate and estimate the rate of convergence for the Bézier variant of the Phillips operators in simultaneous approximations by means of the decomposition technique of functions of bounded variation. We denote the class $B_{r,\beta}$ by

$$B_{r,\beta}=\Big\{f:f^{(r-1)}\in C[0,\infty),\ f_{\pm}^{(r)}(x)\text{ exist everywhere and are bounded on every}$$
$$\text{finite subinterval of }[0,\infty)\text{ and }\ f_{\pm}^{(r)}(x)=O(e^{\beta t})\,(t\to\infty),\ \text{for some }\beta>0\Big\},$$

$r=1,2,\dots$. By $f_{\pm}^{(0)}(x)$ we mean $f(x\pm)$. Our main theorem is stated as:

Theorem 1.1. *Let $f\in B_{r,\beta}$, $r=1,2,...$ and $\beta>0$. Then for every $x\in(0,\infty)$ and $n\geq\max\{r^2+r,\ 4\beta\}$, we have*

$$\left|P_{n,\alpha}^{(r)}(f,x)-\frac{1}{\alpha+1}\left\{f_{+}^{(r)}(x)+\alpha\,f_{-}^{(r)}(x)\right\}\right|\leq\frac{r+\alpha-1}{\sqrt{2enx}}\left|f_{+}^{(r)}(x)-\,f_{-}^{(r)}(x)\right|$$
$$+\frac{1}{n}\left(1+\frac{2\alpha(1+2x)}{x^2}\right)\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_{r,x})+\alpha\frac{\sqrt{2x+1}}{x\sqrt{n}}2^{r/2}e^{2\beta x},$$

where $g_{r,x}$ is the auxiliary function defined by

$$g_{r,x}(t)=\begin{cases}f^{(r)}(t)-f_{+}^{(r)}(x), & x<t<\infty\\ 0, & t=x\\ f^{(r)}(t)-f_{-}^{(r)}(x), & 0\leq t<x\end{cases},$$

$\bigvee_a^b(g_{r,x}(t))$ is the total variation of $g_{r,x}(t)$ on $[a,b]$. In particular $g_{0,x}(t)\equiv g_x(t)$, defined in [4].

2. AUXILIARY RESULTS

In this section we give certain lemmas, which are necessary for proving the main theorem.

Lemma 2.1. *For all $x\in(0,\infty)$, $\alpha\geq1$ and $k\in\mathbb{N}\cup\{0\}$, we have*

$$p_{n,k}(x)\leq\frac{1}{\sqrt{2enx}}$$

and

$$Q_{n,k}^{(\alpha)}(x)\leq\frac{\alpha}{\sqrt{2enx}},$$

where the constant $1/\sqrt{2e}$ and the estimation order $n^{-1/2}$ (for $n\to\infty$) are the best possible.

Lemma 2.2 ([3]). *If* $f \in L_1[0,\infty)$, $f^{(r-1)} \in A.C._{loc}$, $r \in \mathbb{N}$ *and* $f^{(r)} \in L_1[0,\infty)$, *then*

$$P_n^{(r)}(f,x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k+r-1}(t)\, f^{(r)}(t)dt.$$

Lemma 2.3 ([3]). *For* $m \in \mathbb{N}\cup\{0\}$, $r \in \mathbb{N}$, *if we define the m-th order moment by*

$$\mu_{r,n,m}(x) = n\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k+r-1}(t)\,(t-x)^m dt$$

then $\mu_{r,n,0}(x) = 1$, $\mu_{r,n,1}(x) = \frac{r}{n}$ *and* $\mu_{r,n,2}(x) = \frac{2nx+r(r+1)}{n^2}$.

Also there holds the following recurrence relation

$$n\mu_{r,n,m+1}(x) = x[\mu^{(1)}_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)] + (m+r)\mu_{r,n,m}(x).$$

Consequently by the recurrence relation, for all $x \in [0,\infty)$, *we have*

$$\mu_{r,n,m}(x) = O\left(n^{-[(m+1)/2]}\right).$$

Remark 2.4. In particular, by Lemma 2.3, for given any number $n \geq r^2 + r$ and $0 < x < \infty$, we have

$$\mu_{r,n,2}(x) \leq \frac{2x+1}{n}. \tag{2.1}$$

Remark 2.5. We can observe from Lemma 2.2 and Lemma 2.3 that for $r = 0$, the summation over k starts from 1. For $r = 0$, Lemma 2.3 may be defined as [5, Lemma 2], with $c = 0$.

Lemma 2.6. *Suppose* $x \in (0,\infty)$, $r \in \mathbb{N}\cup\{0\}$, $\alpha \geq 1$ *and*

$$K_{r,n,\alpha}(x,t) = n\sum_{k=0}^{\infty} Q^{(\alpha)}_{n,k}(x)p_{n,k+r-1}(t).$$

Then for $n \geq r^2 + r$, *there hold*

$$\int_0^y K_{r,n,\alpha}(x,t)dt \leq \frac{\alpha(2x+1)}{n(x-y)^2}, \qquad 0 \leq y < x, \tag{2.2}$$

$$\int_z^{\infty} K_{r,n,\alpha}(x,t)dt \leq \frac{\alpha(2x+1)}{n(z-x)^2}, \qquad x < z < \infty. \tag{2.3}$$

Proof. We first prove (2.2) as follows:

$$\begin{aligned}\int_0^y K_{r,n,\alpha}(x,t)dt &\leq \int_0^y \frac{(x-t)^2}{(x-y)^2}K_{r,n,\alpha}(x,t)dt \\ &\leq \frac{\alpha}{(x-y)^2}P_n((t-x)^2,x) \\ &\leq \frac{\alpha\mu_{r,n,2}(x)}{(x-y)^2} \leq \frac{\alpha(2x+1)}{n(x-y)^2},\end{aligned}$$

by using (2.1). The proof of (2.3) follows along similar lines. □

3. PROOF

Proof of Theorem 1.1. Clearly

$$\left|P_{n,\alpha}^{(r)}(f,x)-\frac{1}{\alpha+1}\left\{f_{+}^{(r)}(x)+\alpha f_{-}^{(r)}(x)\right\}\right| \le n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_0^{\infty}p_{n+r-1,k}(x)\left|g_{r,x}(t)\right|dt + \frac{1}{\alpha+1}\left|f_{+}^{(r)}(x)-f_{-}^{(r)}(x)\right|\left|P_{n,\alpha}^{(r)}(\operatorname{sgn}_{\alpha}(t-x),x)\right|. \tag{3.1}$$

We first estimate $P_{n,\alpha}^{(r)}(\operatorname{sgn}_{\alpha}(t-x),x)$ as follows:

$$\begin{aligned}
P_{n,\alpha}^{(r)}(\operatorname{sgn}_{\alpha}(t-x),x) &= n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\int_0^{\infty}\alpha p_{n,k+r-1}(t)dt-(1+\alpha)\int_0^{x}p_{n,k+r-1}(t)dt\right)\\
&=\alpha-(1+\alpha)n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_0^{\infty}p_{n,k+r-1}(t)dt\\
&=\alpha-(1+\alpha)n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(1-\sum_{j=0}^{k+r-1}p_{n,j}(x)\right)\\
&=(1+\alpha)n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\sum_{j=0}^{k+r-1}p_{n,j}(x)-1\\
&=(1+\alpha)n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\left(\sum_{j=0}^{k}p_{n,j}(x)+\frac{r-1}{\sqrt{2enx}}\right)-1\\
&=(1+\alpha)\left[\sum_{j=0}^{\infty}p_{n,j}(x)\sum_{k=j}^{\infty}Q_{n,k}^{(\alpha)}(x)+\frac{r-1}{\sqrt{2enx}}\right]-1\\
&=(1+\alpha)\left[\sum_{j=0}^{\infty}p_{n,j}(x)\left[J_{n,j}(x)\right]^{\alpha}+\frac{r-1}{\sqrt{2enx}}\right]-\sum_{j=0}^{\infty}Q_{n,j}^{(\alpha)}(x).
\end{aligned}$$

By the mean value theorem, we find that

$$Q_{n,j}^{(\alpha+1)}(x)=\left[J_{n,j}(x)\right]^{\alpha+1}-\left[J_{n,j+1}(x)\right]^{\alpha+1}=(\alpha+1)p_{n,j}(x)\left[\gamma_{n,j}(x)\right]^{\alpha},$$

where

$$J_{n,j+1}(x)<\gamma_{n,j}(x)<J_{n,j}(x).$$

Hence by Lemma 2.1, we have

$$\begin{aligned}
&\left|P_{n,\alpha}^{(r)}(\operatorname{sgn}_{\alpha}(t-x),x)\right|\\
&\le\left|(1+\alpha)\left[\sum_{j=0}^{\infty}p_{n,j}(x)\left(\left[J_{n,j}(x)\right]^{\alpha}-\left[\gamma_{n,j}(x)\right]^{\alpha}\right)\right]\right|+\frac{(1+\alpha)(r-1)}{\sqrt{2enx}}\\
&\le(1+\alpha)\left[\sum_{j=0}^{\infty}p_{n,j}(x)Q_{n,k}^{(\alpha)}(x)+\frac{r-1}{\sqrt{2enx}}\right]
\end{aligned}$$

$$\leq (1+\alpha)\left[\frac{\alpha}{\sqrt{2enx}}+\frac{r-1}{\sqrt{2enx}}\right]=\frac{(1+\alpha)(r+\alpha-1)}{\sqrt{2enx}}. \tag{3.2}$$

Next we estimate $P_n^{(r)}(g_{r,x},x)$. By the Lebesgue-Stieltjes integral representation, we have

$$\begin{aligned} P_{n,\alpha}^{(r)}(g_{r,x},x) &= \int_0^\infty g_{r,x}(t)K_{r,n,\alpha}(x,t)dt \\ &= \left(\int_{I_1}+\int_{I_2}+\int_{I_3}+\int_{I_4}\right) g_{r,x}(t)K_{r,n,\alpha}(x,t)dt \\ &= R_1+R_2+R_3+R_{4,} \end{aligned} \tag{3.3}$$

say, where $I_1 = [0, x - x/\sqrt{n}]$, $I_2 = [x - x/\sqrt{n}, x + x/\sqrt{n}]$, $I_3 = [x + x/\sqrt{n}, 2x]$ and $I_4 = [2x,\infty)$. Let us define

$$\eta_{r,n,\alpha}(x,t)=\int_0^t K_{r,n,\alpha}(x,u)du.$$

We first estimate R_1. Writing $y = x - x/\sqrt{n}$ and using integration by parts, we have

$$\begin{aligned} R_1 &= \int_0^y g_{r,x}(t)d_t(\eta_{r,n,\alpha}(x,t)) \\ &= g_{r,x}(y)\eta_{r,n,\alpha}(x,y)-\int_0^y \eta_{r,n,\alpha}(x,t)d_t(g_{r,x}(t)). \end{aligned}$$

By Remark 2.4, it follows that

$$\begin{aligned} |R_1| &\leq \bigvee_y^x (g_{r,x})\eta_{r,n,\alpha}(x,y)+\int_0^y \eta_{r,n,\alpha}(x,t)d_t\left(-\bigvee_t^x(g_{r,x})\right) \\ &\leq \bigvee_y^x (g_{r,x})\frac{\alpha(2x+1)}{n(x-y)^2}+\frac{\lambda x}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_{r,x})\right). \end{aligned}$$

Integrating by parts the last term, we have after simple computation,

$$|R_1|\leq \frac{\alpha(2x+1)}{n}\left[\frac{\bigvee_0^x(g_{r,x})}{x^2}+2\int_0^y \frac{\bigvee_t^x(g_{r,x})}{(x-t)^3}dt\right].$$

Now replacing the variable y in the last integral by $x - x/\sqrt{t}$, we get

$$|R_1|\leq \frac{2\alpha(2x+1)}{nx^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_{r,x}). \tag{3.4}$$

Next we estimate R_2. For $t\in[x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|g_{r,x}(t)| = |g_{r,x}(t)-g_{r,x}(x)|\leq \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_{r,x}).$$

Also by the fact that $\int_a^b d_t(\eta_{r,n}(x,t))\leq 1$ for $(a,b)\subset[0,\infty)$, therefore

$$|R_2|\leq \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_{r,x})\leq \frac{1}{n}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_{r,x}). \tag{3.5}$$

To estimate R_3, we take $z = x + x/\sqrt{n}$, thus

$$\begin{aligned} R_3 &= \int_z^{2x} K_{r,n,\alpha}(x,t) g_{r,x}(t)dt \\ &= -\int_z^{2x} g_{r,x}(t) d_t(1-\eta_{r,n,\alpha}(x,t)) \\ &= -g_{r,x}(2x)(1-\eta_{r,n,\alpha}(x,2x)) + g_{r,x}(z)(1-\eta_{r,n,\alpha}(x,z)) \\ &\qquad + \int_z^{2x} (1-\eta_{r,n,\alpha}(x,t)) d_t(g_{r,x}(t)). \end{aligned}$$

Thus arguing similarly as in the estimate of R_1, we obtain

$$|R_3| \le \frac{2\alpha(2x+1)}{nx^2} \sum_{k=1}^{n} \bigvee_{x}^{x+x/\sqrt{k}} (g_{r,x}). \tag{3.6}$$

Finally we estimate R_4 as follows

$$\begin{aligned} |R_4| &= \left| \int_{2x}^{\infty} K_{r,n,\alpha}(x,t) g_{r,x}(t)dt \right| \\ &\le n\alpha \sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} p_{n,k+r-1}(t)\, e^{\beta t} dt \\ &\le \frac{n\alpha}{x} \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r-1}(t)\, e^{\beta t}\, |t-x|\, dt \\ &\le \frac{\alpha}{x} \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r-1}(t)\, (t-x)^2 dt \right)^{\frac{1}{2}} \\ &\qquad \times \left(n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r-1}(t)\, e^{2\beta t} dt \right)^{\frac{1}{2}}. \end{aligned}$$

For the first expression above we use Remark 2.4. To evaluate the second expression, we proceed as follows:

$$n \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k+r-1}(t)\, e^{2\beta t} dt = n \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n^{k+r-1}}{(k+r-1)!} \int_0^{\infty} t^{k+r-1}\, e^{-(n-2\beta)t} dt,$$

$$n \sum_{k=0}^{\infty} p_{n,k}(x) \frac{n^{k+r-1}}{(k+r-1)!} \frac{\Gamma(k+r)}{(n-2\beta)^{k+r}} = \frac{n^r}{(n-2\beta)^r} \sum_{k=0}^{\infty} \left(\frac{n}{n-2\beta}\right)^k p_{n,k}(x),$$

$$\frac{n^r}{(n-2\beta)^r} e^{-nx} \sum_{k=0}^{\infty} \left(\frac{n^2 x}{n-2\beta}\right)^k \frac{1}{k!} = \frac{n^r}{(n-2\beta)^r} e^{2nx\beta/(n-2\beta)} \le 2^r e^{4\beta x}$$

for $n > 4\beta$. Thus

$$
\begin{aligned}
|R_4| &\leq \frac{\alpha}{x}\left(n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k+r-1}(t)\,(t-x)^2dt\right)^{\frac{1}{2}} \\
&\qquad \times\left(n\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k+r-1}(t)\,e^{2\beta t}dt\right)^{\frac{1}{2}} \\
&\leq \frac{\alpha\sqrt{2x+1}}{x\sqrt{n}}2^{r/2}e^{2\beta x}. \qquad (3.7)
\end{aligned}
$$

Combining the estimates of (3.1)-(3.7), we get the required result. □

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