SHARPENING ON MIRCEA'S INEQUALITY

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Abstract: In this paper, by using one of Chen's theorems, combining the method of mathematical analysis and nonlinear algebraic equation system, Mircea's Inequality involving the area, circumradius and inradius of the triangle is sharpened.

Dedicatory: Dedicated to Shi-Chang Shi on the occasion of his 50th birthday.

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1. Introduction and Main Results

Let S be the area, R the circumradius, r the inradius and p the semi-perimeter of a triangle. The following laconic and beautiful inequality is the so-called Mircea inequality in [1]

$$R + \frac{r}{2} > \sqrt{S}.$$

In 1991, D. S. Mitrinović et al. [2] noted a Mircea-type inequality obtained by D.M. Milošević

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S}. \tag{1.1}$$

In [4], L. Carliz and F. Leuenberger strengthened inequality (1.1) as follows (see also [3])

$$R + r \geq \sqrt[4]{3}\sqrt{S}, \tag{1.2}$$

since (1.2) can be written as

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S} + \frac{1}{6}(R - 2r), \tag{1.3}$$

and from the well-known Euler inequality $R \geq 2r$.

The main purpose of this article is to give a generalization of inequalities (1.1) and (1.2) or (1.3).

Theorem 1.1. *If $k \leq k_0$, then for any triangle, we have*

$$R + \frac{r}{2} \geq \frac{5}{6}\sqrt[4]{3}\sqrt{S} + k(R - 2r), \tag{1.4}$$

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where k_0 is the root on the interval $(\frac{11}{20}, \frac{4}{7})$ of the equation

$$2304k^4 - 896k^3 - 2336k^2 - 856k + 1159 = 0. \tag{1.5}$$

The equality in (1.4) is valid if and only if the triangle is isosceles and the of ratio of its sides is $2 : x_0 : x_0$, where x_0 is the positive root of the following equation

$$x^4 + 28x^3 - 120x^2 + 80x - 16 = 0. \tag{1.6}$$

From Theorem 1.1, we can make the following remarks.

Remark 1. k_0 is the best constant which makes (1.4) hold, and $k_0 = 0.5660532114\ldots$.

Remark 2. The function

$$f(k) = R + \frac{r}{2} - \frac{5}{6}\sqrt[4]{3}\sqrt{S} - k(R - 2r)$$

is a monotone increasing function on $(-\infty, k_0]$.

Remark 3. For $k = \frac{1}{2}$ in (1.4), the inequality

$$R + 3r \geq \frac{5}{3}\sqrt[4]{3}\sqrt{S}$$

holds.

Remark 4. $x_0 = 3.079485433\ldots$.

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2. Some Lemmas

In order to prove Theorem 1.1, we require several lemmas.

Lemma 2.1 ([5, 6], see also [12]).

(i) If the homogeneous inequality $p \geq (>) f_1(R, r)$ *holds for any isosceles triangle whose top angle is greater than or equal to* 60°*, then the inequality* $p \geq (>) f_1(R, r)$ *holds for any triangle.*

(ii) If the homogeneous inequality $p \leq (<) f_1(R, r)$ *holds for any isosceles triangle whose top angle is less than or equal to* 60°*, then the inequality* $p \leq (<) f_1(R, r)$ *holds for any triangle.*

Lemma 2.2 ([7]). *Denote*

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

and

$$g(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m.$$

If $a_0 \neq 0$ *or* $b_0 \neq 0$*, then the polynomials* $f(x)$ *and* $g(x)$ *have common roots if and only if*

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-1} & a_n & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_0 & \cdots & \cdots & \cdots & a_n \\ b_0 & b_1 & b_2 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{vmatrix} = 0,$$

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where $R(f, g)$ *is Sylvester's resultant of* $f(x)$ *and* $g(x)$.

Lemma 2.3 ([7, 8]). *For a given polynomial* $f(x)$ *with real coefficients*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

if the number of sign changes of the revised sign list of its discriminant sequence

$$\{D_1(f), D_2(f), \ldots, D_n(f)\}$$

is v, *then, the number of the pairs of distinct conjugate imaginary roots of* $f(x)$ *equals* v. *Furthermore, if the number of non-vanishing members of the revised sign list is* l, *then, the number of the distinct real roots of* $f(x)$ *equals* $l - 2v$.

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3. The Proof of Theorem 1.1

Proof. It is not difficult to see that the form of the inequality (1.4) is equivalent to $p \le (<) f_1(R, r)$ with the known identity $S = rp$. From Lemma 2.1, we easily see that inequality (1.4) holds if and only if this triangle is an isosceles triangle whose top angle is less than or equal to $60°$.

Let $a = 2, b = c = x \ \ (x \ge 2)$, then (1.4) is equivalent to

$$\frac{x^2}{2\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{2(x+1)} \ge \frac{5}{6}\sqrt[4]{3(x^2-1)} + k\left(\frac{x^2}{2\sqrt{x^2-1}} - \frac{2\sqrt{x^2-1}}{x+1}\right),$$

or

$$x^2 + x - 1 \ge \frac{5}{3}\sqrt[4]{3(x^2-1)^3} + k(x-2)^2. \tag{3.1}$$

For $x = 2$, (3.1) obviously holds. If $x > 2$, then (3.1) is equivalent to

$$k \le \frac{x^2 + x - 1 - \frac{5}{3}\sqrt[4]{3(x^2-1)^3}}{(x-2)^2}.$$

Define a function

$$g(x) = \frac{x^2 + x - 1 - \frac{5}{3}\sqrt[4]{3(x^2-1)^3}}{(x-2)^2} \qquad (x > 2).$$

Calculating the derivative for $g(x)$, we get

$$g'(x) = \frac{5\left[\sqrt[4]{3}(x^2+6x-4) - 6x\sqrt[4]{x^2-1}\right]}{6(x-2)^3\sqrt[4]{x^2-1}}.$$

Let $g'(x) = 0$, we obtain

$$\sqrt[4]{3}(x^2+6x-4) - 6x\sqrt[4]{x^2-1} = 0. \tag{3.2}$$

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It is easy to see that the roots of equation (3.2) must be the roots of the following equation

$$(x^4+28x^3-120x^2+80x-16)(x+2)(x-2)^3=0.$$

For the range of roots of equation (3.2) on $(2,+\infty)$, the roots of equation (3.2) must be the roots of equation (1.6).

It shows that equation (1.6) has only one positive real root on the open interval $(2,+\infty)$. Let x_0 be the positive real root of equation (1.6). Then $x_0 = 3.079485433\ldots$, and

$$\begin{aligned} g(x)_{\min} = g(x_0) &= \frac{x_0^2+x_0-1-\frac{5}{3}\sqrt[4]{3(x_0^2-1)^3}}{(x_0-2)^2} \\ &= 0.5660532114\cdots \in \left(\frac{11}{20},\frac{4}{7}\right). \end{aligned} \tag{3.3}$$

Therefore, the maximum of k is $g(x_0)$.

Now we prove that $g(x_0)$ is the root of equation (1.5).

Consider the nonlinear algebraic equation system as follows

$$\begin{cases} x_0^4+28x_0^3-120x_0^2+80x_0-16=0 \\ u_0^4-3(x_0^2-1)^3=0 \\ x_0^2+x_0-1-\frac{5}{3}u_0-(x_0-2)^2t=0 \end{cases}, \tag{3.4}$$

or

$$\begin{cases} F(x_0)=0 \\ G(x_0)=0 \end{cases}, \tag{3.5}$$

where

$$F(x_0)=x_0^4+28x_0^3-120x_0^2+80x_0-16,$$

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and

$$\begin{aligned}G(x_0) = {} & 81\,(-1+t)^4\,{x_0}^8 - 324\,(1+4\,t)\,(-1+t)^3\,{x_0}^7 \\ & + \left(-1713 + 2592\,t + 3402\,t^2 - 15228\,t^3 + 9072\,t^4\right){x_0}^6 \\ & - 324\,(-2+7\,t)\,(-1+t)\,(1+4\,t)^2\,{x_0}^5 \\ & + \left(5220 - 6480\,t - 26730\,t^2 - 6480\,t^3 + 90720\,t^4\right){x_0}^4 \\ & - 324\,(-2+7\,t)\,(1+4\,t)^3\,{x_0}^3 \\ & + \left(-5463 + 4212\,t + 34992\,t^2 + 119232\,t^3 + 145152\,t^4\right){x_0}^2 \\ & - 324\,(1+4\,t)^4\,x_0 + 1956 + 1296\,t + 7776\,t^2 + 20736\,t^3 + 20736\,t^4.\end{aligned}$$

We have that $g(x_0)$ is also the solution of the nonlinear algebraic equation system (3.4) or (3.5). From Lemma 2.2, we get

$$R(F,G) = 44079842304p_1(t)p_2(t)p_3(t) = 0,$$

where

$$p_1(t) = 2304t^4 - 896t^3 - 2336t^2 - 856t + 1159,$$
$$p_2(t) = 2304t^4 - 46976t^3 + 51104t^2 - 35496t + 10939,$$

$$\begin{aligned}p_3(t) = {} & 1327104t^8 - 27574272t^7 + 270856192t^6 - 218763264t^5 - 111704320t^4 \\ & + 78507776t^3 + 170893152t^2 - 164410112t + 62195869.\end{aligned}$$

The revised sign list of the discriminant sequence of $p_2(t)$ is

$$[1, 1, -1, -1]. \tag{3.6}$$

The revised sign list of the discriminant sequence of $p_3(t)$ is

$$[1, -1, -1, -1, 1, -1, 1, 1]. \tag{3.7}$$

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So the number of the sign changes of the revised sign list of (3.6) equals 1, then with Lemma 2.2, the equation $p_2(t) = 0$ has 2 distinct real roots. And by using the function "realroot()"[10, 11] in Maple 9.0, we can find that $p_2(t) = 0$ has 2 distinct real roots in the following intervals

$$\left[\frac{1}{2}, \frac{17}{32}\right], \quad \left[\frac{77}{4}, \frac{617}{32}\right]$$

and no real root on the interval $(\frac{11}{20}, \frac{4}{7})$.

If the number of the sign changes of the revised sign list of (3.7) equals 4, then from Lemma 2.3, the equation $p_3(t) = 0$ has 4 pairs distinct conjugate imaginary roots. That is to say, $p_3(t) = 0$ has no real root.

From (3.3), we easily deduce that $g(x_0)$ is the root of the equation $p_1(t) = 0$. Namely, $g(x_0)$ is the root of equation (1.5).

Further, considering the proof above, we can easily obtain the required result in (1.4).

Thus, the proof of Theorem 1.1 is completed. $\square$

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