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APPLICATIONS OF DIFFERENTIAL SUBORDINATION TO CERTAIN SUBCLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS

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ABSTRACT. By making use of the principle of differential subordination, the authors investigate several inclusion relationships and other interesting properties of certain subclasses of meromorphically multivalent functions which are defined here by means of a linear operator. They also indicate relevant connections of the various results presented in this paper with those obtained in earlier works.

Key words and phrases: Meromorphic functions, Differential subordination, Hadamard product (or convolution), Multivalent functions, Linear operator, Hypergeometric function.

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1. INTRODUCTION AND DEFINITIONS

For any integer $m > -p$, let $\Sigma_{p,m}$ denote the class of all meromorphic functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}.$$

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For convenience, we write

$$\Sigma_{p,-p+1} = \Sigma_p.$$

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad \text{in} \quad \mathbb{U} \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(w(z)\big) \ (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, we have the following equivalence (cf., e.g., [5]; see also [6, p. 4]):

$$f(z) \prec g(z) \ (z \in \mathbb{U}) \qquad \Longleftrightarrow \qquad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions $f(z) \in \Sigma_{p,m}$, given by (1.1), and $g(z) \in \Sigma_{p,m}$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \quad (m > -p;\ p \in \mathbb{N}), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f \star g)(z) := z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k =: (g \star f)(z) \quad (m > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}). \tag{1.3}$$

Following the recent work of Liu and Srivastava [3], for a function $f(z)$ in the class $\Sigma_{p,m}$, given by (1.1), we now define a linear operator D^n by

$$D^0 f(z) = f(z),$$

$$D^1 f(z) = z^{-p} + \sum_{k=m}^{\infty} (p+k+1) a_k z^k = \frac{\big(z^{p+1} f(z)\big)'}{z^p},$$

and (in general)

$$\begin{aligned} D^n f(z) = D\big(D^{n-1} f(z)\big) &= z^{-p} + \sum_{k=m}^{\infty} (p+k+1)^n a_k z^k \\ &= \frac{\big(z^{p+1} D^{n-1} f(z)\big)'}{z^p} \quad (n \in \mathbb{N}). \end{aligned} \tag{1.4}$$

It is easily verified from (1.4) that

$$z\big(D^n f(z)\big)' = D^{n+1} f(z) - (p+1) D^n f(z) \quad \Big(f \in \Sigma_{p,m};\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\Big). \tag{1.5}$$

The case $m = 0$ of the linear operator D^n was introduced recently by Liu and Srivastava [3], who investigated (among other things) several inclusion relationships involving various subclasses of meromorphically p-valent functions, which they defined by means of the linear operator D^n (see also [2]). A special case of the linear operator D^n for $p = 1$ and $m = 0$ was considered earlier by Uralegaddi and Somanatha [13]. Aouf and Hossen [1] also obtained several results involving the operator D^n for $m = 0$ and $p \in \mathbb{N}$.

Making use of the principle of differential subordination as well as the linear operator D^n, we now introduce a subclass of the function class $\Sigma_{p,m}$ as follows.

Definition. For fixed parameters A and B $(-1 \leqq B < A \leqq 1)$, we say that a function $f(z) \in \Sigma_{p,m}$ is in the class $\Sigma_{p,m}^{n}(A,B)$, if it satisfies the following subordination condition:

(1.6)
$$-\frac{z^{p+1}\big(D^n f(z)\big)'}{p} \prec \frac{1+Az}{1+Bz} \quad (n \in \mathbb{N}_0;\ z \in \mathbb{U}).$$

In view of the definition of differential subordination, (1.6) is equivalent to the following condition:
$$\left|\frac{z^{p+1}\big(D^n f(z)\big)' + p}{Bz^{p+1}\big(D^n f(z)\big)' + pA}\right| < 1 \quad (z \in \mathbb{U}).$$
For convenience, we write
$$\Sigma_{p,m}^{n}\left(1-\frac{2\alpha}{p}, -1\right) = \Sigma_{p,m}^{n}\left(\alpha\right),$$
where $\Sigma_{p,m}^{n}(\alpha)$ denotes the class of functions in $\Sigma_{p,m}$ satisfying the following inequality:
$$\Re\big(-z^{p+1}\big(D^n f(z)\big)'\big) > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}).$$
In particular, we have
$$\Sigma_{p,0}^{n}\left(A,B\right) = \mathcal{R}_{n,p}(A,B),$$
where $\mathcal{R}_{n,p}(A,B)$ is the function class introduced and studied by Liu and Srivastava [3]. The function class $\mathcal{H}(p;A,B)$, considered by Mogra [7], happens to be a *further* special case of the Liu-Srivastava class $\mathcal{R}_{n,p}(A,B)$ when $n = 0$.

In the present paper, we derive several inclusion relationships for the function class $\Sigma_{p,m}^{n}(A,B)$ and investigate various other properties of functions belonging to the class $\Sigma_{p,m}^{n}(A,B)$, which we have defined here by means of the linear operator D^n. These include (for example) some mapping properties involving the linear operator D^n. Relevant connections of the results presented in this paper with those obtained in earlier works are also pointed out.

2. PRELIMINARY LEMMAS

In proving our main results, we need each of the following lemmas.

Lemma 1 (Miller and Mocanu [5]; see also [6])**.** *Let the function* $h(z)$ *be analytic and convex* (*univalent*) *in* $\mathbb{U}$ *with* $h(0) = 1$. *Suppose also that the function* $\phi(z)$ *given by*

(2.1)
$$\phi(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \cdots$$

is analytic in $\mathbb{U}$. *If*

(2.2)
$$\phi(z) + \frac{z\phi'(z)}{\gamma} \prec h(z) \quad \big(\Re(\gamma) \geqq 0;\ \gamma \neq 0;\ z \in \mathbb{U}\big),$$

then
$$\phi(z) \prec \psi(z) = \frac{\gamma}{p+m}\, z^{-\frac{\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1}\, h(t)dt \prec h(z) \quad (z \in \mathbb{U}),$$
and $\psi(z)$ *is the best dominant of* (2.2).

With a view to stating a well-known result (Lemma 2 below), we denote by $\mathcal{P}(\gamma)$ the class of functions $\varphi(z)$ given by

(2.3)
$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \cdots,$$

which are analytic in $\mathbb{U}$ and satisfy the following inequality:
$$\Re\big(\varphi(z)\big) > \gamma \quad (0 \leqq \gamma < 1;\ z \in \mathbb{U}).$$

Lemma 2 (cf., e.g., Pashkouleva [9]). *Let the function $\varphi(z)$, given by (2.3), be in the class $\mathcal{P}(\gamma)$. Then*

$$\Re\{\varphi(z)\} \geqq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \geqq \gamma < 1;\ z \in \mathbb{U}).$$

Lemma 3 (see [12]). *For $0 \leqq \gamma_1, \gamma_2 < 1$,*

$$\mathcal{P}(\gamma_1) \star \mathcal{P}(\gamma_2) \subset \mathcal{P}(\gamma_3) \quad \big(\gamma_3 := 1 - 2(1-\gamma_1)(1-\gamma_2)\big).$$

The result is the best possible.

For real or complex numbers a, b, and c $(c \notin \mathbb{Z}_0^- := \{0, -1, -2, \cdots\})$, the *Gauss hypergeometric function* is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots .$$

We note that the above series converges absolutely for $z \in \mathbb{U}$ and hence represents an analytic function in $\mathbb{U}$ (see, for details, [14, Chapter 14]).

Each of the identities (asserted by Lemma 4 below) is well-known (cf., e.g., [14, Chapter 14]).

Lemma 4. *For real or complex parameters a, b, and c $(c \notin \mathbb{Z}_0^-)$,*

$$\begin{aligned} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, dt \\ = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z) \quad \big(\Re(c) > \Re(b) > 0\big); \end{aligned} \tag{2.4}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a}\, {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right); \tag{2.5}$$

$${}_2F_1(a, b; c; z) = {}_2F_1(a, b-1; c; z) + \frac{az}{c}\, {}_2F_1(a+1, b; c+1; z); \tag{2.6}$$

$${}_2F_1\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\, \Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right) \Gamma\left(\frac{b+1}{2}\right)}. \tag{2.7}$$

We now recall a result due to Singh and Singh [11] as Lemma 5 below.

Lemma 5. *Let $\Phi(z)$ be analytic in $\mathbb{U}$ with*

$$\Phi(0) = 1 \quad \textit{and} \quad \Re\big(\Phi(z)\big) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Then, for any function $F(z)$ analytic in $\mathbb{U}$, $(\Phi \star F)(\mathbb{U})$ is contained in the convex hull of $F(\mathbb{U})$.

3. THE MAIN SUBORDINATION THEOREMS AND THE ASSOCIATED FUNCTIONAL INEQUALITIES

Unless otherwise mentioned, we shall assume *throughout the sequel* that m is an integer greater than $-p$, and that

$$-1 \leqq B < A \leqq 1,\ \lambda > 0,\ n \in \mathbb{N}_0,\ \text{and}\ \ p \in \mathbb{N}.$$

Theorem 1. *Let the function* $f(z)$ *defined by* (1.1) *satisfy the following subordination condition*:

$$-\frac{(1-\lambda)z^{p+1}\big(D^n f(z)\big)' + \lambda z^{p+1}\big(D^{n+1} f(z)\big)'}{p} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Then

$$-\frac{z^{p+1}\big(D^n f(z)\big)'}{p} \prec Q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{3.1}$$

where the function $Q(z)$ *given by*

$$Q(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1, 1; \dfrac{1}{\lambda(p+m)} + 1; \dfrac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \dfrac{A}{\lambda(p+m)+1}\ z & (B = 0) \end{cases}$$

is the best dominant of (3.1). *Furthermore*,

$$\Re\left(-\frac{z^{p+1}\big(D^n f(z)\big)'}{p}\right) > \rho \quad (z \in \mathbb{U}), \tag{3.2}$$

where

$$\rho = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1, 1; \dfrac{1}{\lambda(p+m)} + 1; \dfrac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{A}{\lambda(p+m)+1} & (B = 0). \end{cases}$$

The inequality in (3.2) *is the best possible.*

Proof. Consider the function $\phi(z)$ defined by

$$\phi(z) = -\frac{z^{p+1}\big(D^n f(z)\big)'}{p} \quad (z \in \mathbb{U}). \tag{3.3}$$

Then $\phi(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Applying the identity (1.5) in (3.3) and differentiating the resulting equation with respect to z, we get

$$-\frac{(1-\lambda)z^{p+1}\big(D^n f(z)\big)' + \lambda z^{p+1}\big(D^{n+1} f(z)\big)'}{p} = \phi(z) + \lambda\, z\phi'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Now, by using Lemma 1 for $\gamma = 1/\lambda$, we deduce that

$$-\frac{z^{p+1}\big(D^n f(z)\big)'}{p} \prec Q(z)$$
$$= \frac{1}{\lambda(p+m)} z^{-\left\{1/\lambda(p+m)\right\}} \int_0^z t^{\left\{1/\lambda(p+m)\right\}-1} \left(\frac{1+At}{1+Bt}\right) dt$$
$$= \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1} \, {}_2F_1\left(1,1;\dfrac{1}{\lambda(p+m)}+1;\dfrac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \dfrac{A}{\lambda(p+m)+1}\, z & (B = 0), \end{cases}$$

by change of variables followed by the use of the identities (2.4), (2.5), and (2.6) (*with* $b = 1$ and $c = a + 1$). This proves the assertion (3.1) of Theorem 1.

Next, in order to prove the assertion (3.2) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \left\{\Re\big(Q(z)\big)\right\} = Q(-1). \tag{3.4}$$

Indeed, for $|z| \leqq r < 1$,

$$\Re\left(\frac{1+Az}{1+Bz}\right) \geqq \frac{1-Ar}{1-Br} \quad (|z| \leqq r < 1)).$$

Upon setting

$$\mathcal{G}(s,z) = \frac{1+Asz}{1+Bsz} \quad \text{and} \quad d\nu(s) = \frac{1}{\lambda(p+m)}\, s^{\{1/\lambda(p+m)\}-1}\, ds \quad (0 \leqq s \leqq 1),$$

which is a positive measure on the closed interval $[0,1]$, we get

$$Q(z) = \int_0^1 \mathcal{G}(s,z)\, d\nu(s),$$

so that

$$\Re\big(Q(z)\big) \geqq \int_0^1 \left(\frac{1-Asr}{1-Bsr}\right) d\nu(s) = Q(-r) \quad (|z| \leqq r < 1).$$

Letting $r \to 1-$ in the above inequality, we obtain the assertion (3.2) of Theorem 1.

Finally, the estimate in (3.2) is the best possible as the function $Q(z)$ is the best dominant of (3.1). □

For $\lambda = 1$ and $m = 0$, Theorem 1 yields the following result which improves the corresponding work of Liu and Srivastava [3, Theorem 1].

Corollary 1. *The following inclusion property holds true for the function class* $\mathcal{R}_{n,p}(A,B)$:

$$\mathcal{R}_{n+1,p}(A,B) \subset \mathcal{R}_{n,p}(1-2\varrho,-1) \subset \mathcal{R}_{n,p}(A,B),$$

where

$$\varrho = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1} \, {}_2F_1\left(1,1;\dfrac{1}{p}+1;\dfrac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{A}{p+1} & (B = 0). \end{cases}$$

The result is the best possible.

Putting

$$A = 1 - \frac{2\alpha}{p},\ B = -1,\ \lambda = 1,\ n = 0, \quad \text{and} \quad m = -p + 2$$

in Theorem 1, we get Corollary 2 below.

Corollary 2. *If $f(z) \in \Sigma_{p,-p+2}$ satisfies the following inequality*:

$$\Re\Big(- z^{p+1}\Big\{(p+2)f'(z) + zf''(z)\Big\}\Big) > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}),$$

then

$$\Re\big(- z^{p+1}f'(z)\big) > \alpha + (p-\alpha)\Big(\frac{\pi}{2} - 1\Big) \quad (z \in \mathbb{U}).$$

The result is the best possible.

Remark 1. *From Corollary 2, we note that, if $f(z) \in \Sigma_{p,-p+2}$ satisfies the following inequality:*

$$\Re\Big(- z^{p+1}\left\{(p+2)f'(z) + zf''(z)\right\}\Big) > -\frac{p(\pi-2)}{4-\pi} \quad (z \in \mathbb{U}),$$

then

$$\Re\big(- z^{p+1}f'(z)\big) > 0 \quad (z \in \mathbb{U}).$$

This result is the best possible.

The result (asserted by Remark 1 above) was also obtained by Pap [8].

Theorem 2. *If $f(z) \in \Sigma_{p,m}^{n}(\alpha)$ $(0 \leqq \alpha < p)$, then*

$$\Re\Big(- z^{p+1}\Big\{(1-\lambda)\big(D^n f(z)\big)' + \lambda\big(D^{n+1} f(z)\big)'\Big\}\Big) > \alpha \quad (|z| < R), \tag{3.5}$$

where

$$R = \left(\sqrt{1 + \lambda^2(p+m)^2} - \lambda(p+m)\right)^{\frac{1}{p+m}}.$$

The result is the best possible.

Proof. We begin by writing

$$-z^{p+1}\big(D^n f(z)\big)' = \alpha + (p-\alpha)u(z) \quad (z \in \mathbb{U}). \tag{3.6}$$

Then, clearly, $u(z)$ is of the form (2.1), is analytic in $\mathbb{U}$, and has a positive real part in $\mathbb{U}$. Making use of the identity (1.5) in (3.6) and differentiating the resulting equation with respect to z, we observe that

$$-\frac{z^{p+1}\Big((1-\lambda)\big(D^n f(z)\big)' + \lambda\big(D^{n+1} f(z)\big)'\Big) + \alpha}{p-\alpha} = u(z) + \lambda z u'(z). \tag{3.7}$$

Now, by applying the following estimate [4]:

$$\frac{|zu'(z)|}{\Re\{u(z)\}} \leqq \frac{2(p+m)r^{p+m}}{1 - r^{2(p+m)}} \quad (|z| = r < 1)$$

in (3.7), we get

$$\Re\left(-\frac{z^{p+1}\Big((1-\lambda)\big(D^n f(z)\big)' + \lambda\big(D^{n+1} f(z)\big)'\Big) + \alpha}{p-\alpha} \right) \geqq \Re\big(u(z)\big) \cdot \left(1 - \frac{2\lambda(p+m)r^{p+m}}{1 - r^{2(p+m)}}\right). \tag{3.8}$$

It is easily seen that the right-hand side of (3.8) is positive, provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (3.5) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f(z) \in \Sigma_{p,m}$ defined by

$$-z^{p+1}\big(D^n f(z)\big)' = \alpha + (p-\alpha)\,\frac{1+z^{p+m}}{1-z^{p+m}} \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}).$$

Noting that

$$-\,\frac{z^{p+1}\Big((1-\lambda)\big(D^n f(z)\big)' + \lambda\big(D^{n+1} f(z)\big)'\Big) + \alpha}{p-\alpha} = \frac{1 - z^{2(p+m)} + 2\lambda(p+m)z^{p+m}}{(1-z)^{2(p+m)}} = 0$$

for

$$z = R \cdot \exp\left(\frac{i\pi}{p+m}\right),$$

we complete the proof of Theorem 2. □

Putting $\lambda = 1$ in Theorem 2, we deduce the following result.

Corollary 3. *If $f(z) \in \Sigma^n_{p,m}(\alpha)$ $(0 \leqq \alpha < p)$, then $f(z) \in \Sigma^{n+1}_{p,m}(\alpha)$ for $|z| < \widetilde{R}$, where*

$$\widetilde{R} = \left(\sqrt{1+(p+m)^2} - (p+m)\right)^{\frac{1}{p+m}}.$$

The result is the best possible.

Theorem 3. *Let $f(z) \in \Sigma^n_{p,m}(A,B)$ and let*

$$\mathcal{F}_{\delta,p}(f)(z) = \frac{\delta}{z^{\delta+p}} \int_0^z t^{\delta+p-1} f(t)\,dt \quad (\delta > 0;\ z \in \mathbb{U}^*). \tag{3.9}$$

Then

$$-\frac{z^{p+1}\big(D^n \mathcal{F}_{\delta,p}(f)(z)\big)'}{p} \prec \Theta(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{3.10}$$

where the function $\Theta(z)$ given by

$$\Theta(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}\ {}_2F_1\left(1,1;\dfrac{\delta}{p+m}+1;\dfrac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \dfrac{A\delta}{\delta+p+m}\,z & (B = 0) \end{cases}$$

is the best dominant of (3.10). *Furthermore*,

$$\Re\left(-\frac{z^{p+1}\big(D^n \mathcal{F}_{\delta,p}(f)(z)\big)'}{p}\right) > \varkappa \quad (z \in \mathbb{U}), \tag{3.11}$$

where

$$\varkappa = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\dfrac{\delta}{p+m}+1;\dfrac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{A\delta}{\delta+p+m} & (B = 0). \end{cases}$$

The result is the best possible.

Proof. Setting

$$\phi(z) = -\frac{z^{p+1}\big(D^n\mathcal{F}_{\delta,p}(f)(z)\big)'}{p} \quad (z \in \mathbb{U}), \tag{3.12}$$

we note that $\phi(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Using the following operator identity:

$$z\big(D^n\mathcal{F}_{\delta,p}(f)(z)\big)' = \delta\, D^n f(z) - (\delta+p)\, D^n\mathcal{F}_{\delta,p}(f)(z) \tag{3.13}$$

in (3.12), and differentiating the resulting equation with respect to z, we find that

$$-\frac{z^{p+1}\big(D^n f(z)\big)'}{p} = \phi(z) + \frac{z\phi'(z)}{\delta} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. □

Setting $m = 0$ in Theorem 3, we obtain the following result which improves the corresponding work of Liu and Srivastava [3, Theorem 2].

Corollary 4. *If $\delta > 0$ and $f(z) \in \mathcal{R}_{n,p}(A,B)$, then*

$$\mathcal{F}_{\delta,p}(f)(z) \in \mathcal{R}_{n,p}(1-2\xi,-1) \subset \mathcal{R}_{n,p}(A,B),$$

where

$$\xi = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1-B)^{-1}\ {}_2F_1\left(1,1;\dfrac{\delta}{p}+1;\dfrac{B}{B-1}\right) & (B \neq 0) \\ 1 - \dfrac{A\delta}{\delta+p} & (B=0). \end{cases}$$

The result is the best possible.

Remark 2. *By observing that*

$$z^{p+1}\big(D^n\mathcal{F}_{\delta,p}(f)(z)\big)' = \frac{\delta}{z^\delta}\int_0^z t^{\delta+p}\big(D^n f(t)\big)' dt \quad (f \in \Sigma_{p,m};\ z \in \mathbb{U}), \tag{3.14}$$

Corollary 4 can be restated as follows.

If $\delta > 0$ and $f(z) \in \mathcal{R}_{n,p}(A,B)$, then

$$\Re\left(-\frac{\delta}{pz^\delta}\int_0^z t^{\delta+p}\big(D^n f(t)\big)' dt\right) > \xi \quad (z \in \mathbb{U}),$$

where ξ is given as in Corollary 4.

In view of (3.14), Theorem 3 for

$$A = 1 - \frac{2\alpha}{p},\ B=-1, \text{ and } n = 0$$

yields

Corollary 5. *If $\delta > 0$ and if $f(z) \in \Sigma_{p,m}$ satisfies the following inequality:*

$$\Re\Big(-z^{p+1}f'(z)\Big) > \alpha \quad (0 \leqq \alpha < p;\ z \in \mathbb{U}),$$

then

$$\Re\left(-\frac{\delta}{z^\delta}\int_0^z t^{\delta+p} f'(t)dt\right) > \alpha + (p-\alpha)\left[{}_2F_1\left(1,1;\frac{\delta}{p+m}+1;\frac{1}{2}\right)-1\right] \quad (z \in \mathbb{U}).$$

The result is the best possible.

Theorem 4. *Let $f(z) \in \Sigma_{p,m}$. Suppose also that $g(z) \in \Sigma_{p,m}$ satisfies the following inequality*:

$$\Re\big(z^p D^n g(z)\big) > 0 \quad (z \in \mathbb{U}).$$

If

$$\left| \frac{D^n f(z)}{D^n g(z)} - 1 \right| < 1 \quad (n \in \mathbb{N}_0;\ z \in \mathbb{U}),$$

then

$$\Re\left(-\frac{z\big(D^n f(z)\big)'}{D^n f(z)} \right) > 0 \quad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{9(p+m)^2 + 4p(2p+m)} - 3(p+m)}{2(2p+m)}.$$

Proof. Letting

$$w(z) = \frac{D^n f(z)}{D^n g(z)} - 1 = \kappa_{p+m} z^{p+m} + \kappa_{p+m+1} z^{p+m+1} + \cdots, \tag{3.15}$$

we note that $w(z)$ is analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad \big|w(z)\big| \leqq |z|^{p+m} \quad (z \in \mathbb{U}).$$

Then, by applying the familiar Schwarz lemma, we get

$$w(z) = z^{p+m}\Psi(z),$$

where the function $\Psi(z)$ is analytic in $\mathbb{U}$ and

$$\big|\Psi(z)\big| \leqq 1 \quad (z \in \mathbb{U}).$$

Therefore, (3.15) leads us to

$$D^n f(z) = D^n g(z)\big(1 + z^{p+m}\Psi(z)\big) \quad (z \in \mathbb{U}). \tag{3.16}$$

Making use of logarithmic differentiation in (3.16), we obtain

$$\frac{z\big(D^n f(z)\big)'}{D^n f(z)} = \frac{z\big(D^n g(z)\big)'}{D^n g(z)} + \frac{z^{p+m}\big\{(p+m)\Psi(z) + z\Psi'(z)\big\}}{1 + z^{p+m}\Psi(z)}. \tag{3.17}$$

Setting $\phi(z) = z^p D^n g(z)$, we see that the function $\phi(z)$ is of the form (2.1), is analytic in $\mathbb{U}$,

$$\Re(\phi(z)) > 0 \quad (z \in \mathbb{U}),$$

and

$$\frac{z\big(D^n g(z)\big)'}{D^n g(z)} = \frac{z\phi'(z)}{\phi(z)} - p,$$

so that we find from (3.17) that

$$\Re\left(-\frac{z\big(D^n f(z)\big)'}{D^n f(z)} \right) \geqq p - \left| \frac{z\phi'(z)}{\phi(z)} \right| - \left| \frac{z^{p+m}\big\{(p+m)\Psi(z) + z\Psi'(z)\big\}}{1 + z^{p+m}\Psi(z)} \right| \quad (z \in \mathbb{U}). \tag{3.18}$$

Now, by using the following known estimates [10] (see also [4]):

$$\left| \frac{\phi'(z)}{\phi(z)} \right| \leqq \frac{2(p+m)r^{p+m-1}}{1 - r^{2(p+m)}} \quad \text{and} \quad \left| \frac{(p+m)\Psi(z) + z\Psi'(z)}{1 + z^{p+m}\Psi(z)} \right| \leqq \frac{(p+m)}{1 - r^{p+m}} \quad (|z| = r < 1)$$

in (3.18), we obtain

$$\Re\left(-\frac{z\big(D^n f(z)\big)'}{D^n f(z)} \right) \geqq \frac{p - 3(p+m)r^{p+m} - (2p+m)r^{2(p+m)}}{1 - r^{2(p+m)}} \quad (|z| = r < 1),$$

which is certainly positive, provided that $r < R_0$, R_0 being given as in Theorem 4. □

Theorem 5. *Let* $-1 \leqq B_j < A_j \leqq 1$ $(j = 1, 2)$. *If each of the functions* $f_j(z) \in \Sigma_p$ *satisfies the following subordination condition*:

$$(1-\lambda)z^p D^n f_j(z) + \lambda z^p D^{n+1} f_j(z) \prec \frac{1 + A_j z}{1 + B_j z} \quad (j = 1, 2;\ z \in \mathbb{U}), \tag{3.19}$$

then

$$(1-\lambda)z^p D^n H(z) + \lambda z^p D^{n+1} H(z) \prec \frac{1 + (1 - 2\eta)z}{1 - z} \quad (z \in \mathbb{U}), \tag{3.20}$$

where

$$H(z) = D^n(f_1 \star f_2)(z)$$

and

$$\eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[1 - \frac{1}{2}\, {}_2F_1\left(1, 1; \frac{1}{\lambda} + 1; \frac{1}{2}\right)\right].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Suppose that each of the functions $f_j(z) \in \Sigma_p$ $(j = 1, 2)$ satisfies the condition (3.19). Then, by letting

$$\varphi_j(z) = (1-\lambda)z^p D^n f_j(z) + \lambda z^p D^{n+1} f_j(z) \quad (j = 1, 2), \tag{3.21}$$

we have

$$\varphi_j(z) \in \mathcal{P}(\gamma_j) \quad \left(\gamma_j = \frac{1 - A_j}{1 - B_j};\ j = 1, 2\right).$$

By making use of the operator identity (1.5) in (3.21), we observe that

$$D^n f_j(z) = \frac{1}{\lambda}\, z^{-p-(1/\lambda)} \int_0^z t^{(1/\lambda)-1}\, \varphi_j(t)dt \quad (j = 1, 2),$$

which, in view of the definition of $H(z)$ given already with (3.20), yields

$$D^n H(z) = \frac{1}{\lambda}\, z^{-p-(1/\lambda)} \int_0^z t^{(1/\lambda)-1}\, \varphi_0(t)dt, \tag{3.22}$$

where, for convenience,

$$\begin{aligned} \varphi_0(z) &= (1-\lambda)z^p D^n H(z) + \lambda z^p D^{n+1} H(z) \\ &= \frac{1}{\lambda}\, z^{-(1/\lambda)} \int_0^z t^{(1/\lambda)-1}\, (\varphi_1 \star \varphi_2)(t)dt. \end{aligned} \tag{3.23}$$

Since $\varphi_1(z) \in \mathcal{P}(\gamma_1)$ and $\varphi_2(z) \in \mathcal{P}(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 \star \varphi_2)(z) \in \mathcal{P}(\gamma_3) \quad \big(\gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2)\big). \tag{3.24}$$

Now, by using (3.24) in (3.23) and then appealing to Lemma 2 and Lemma 4, we get

$$\begin{aligned}\Re\{\varphi_0(z)\} &= \frac{1}{\lambda}\int_0^1 u^{(1/\lambda)-1}\,\Re\{(\varphi_1\star\varphi_2)\}(uz)du\\ &\geqq \frac{1}{\lambda}\int_0^1 u^{(1/\lambda)-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u|z|}\right)du\\ &> \frac{1}{\lambda}\int_0^1 u^{(1/\lambda)-1}\left(2\gamma_3-1+\frac{2(1-\gamma_3)}{1+u}\right)du\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\frac{1}{\lambda}\int_0^1 u^{(1/\lambda)-1}(1+u)^{-1}du\right)\\ &= 1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left[1-\frac{1}{2}\,{}_2F_1\left(1,1;\frac{1}{\lambda}+1;\frac{1}{2}\right)\right]\\ &= \eta \quad (z\in\mathbb{U}).\end{aligned}$$

When $B_1 = B_2 = -1$, we consider the functions $f_j(z)\in\Sigma_p$ $(j=1,2)$, which satisfy the hypothesis (3.19) of Theorem 5 and are defined by

$$D^n f_j(z) = \frac{1}{\lambda}\,z^{-(1/\lambda)}\int_0^z t^{(1/\lambda)-1}\left(\frac{1+A_j t}{1-z}\right)dt \quad (j=1,2).$$

Thus it follows from (3.23) and Lemma 4 that

$$\begin{aligned}\varphi_0(z) &= \frac{1}{\lambda}\int_0^1 u^{(1/\lambda)-1}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right)du\\ &= 1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}\,{}_2F_1\left(1,1;\frac{1}{\lambda}+1;\frac{z}{z-1}\right)\\ &\longrightarrow 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2)\,{}_2F_1\left(1,1;\frac{1}{\lambda}+1;\frac{1}{2}\right) \quad \text{as} \quad z\to -1,\end{aligned}$$

which evidently completes the proof of Theorem 5. □

By setting

$$A_j = 1-2\alpha_j,\ B_j=-1\ (j=1,2), \quad \text{and} \quad n=0$$

in Theorem 5, we obtain the following result which refines the work of Yang [15, Theorem 4].

Corollary 6. *If the functions $f_j(z)\in\Sigma_p$ $(j=1,2)$ satisfy the following inequality*:

$$\Re\Big((1+\lambda p)z^p f_j(z)+\lambda z^{p+1}f_j'(z)\Big) > \alpha_j \quad (0\leqq\alpha_j<1;\ j=1,2;\ z\in\mathbb{U}), \tag{3.25}$$

then

$$\Re\Big((1+\lambda p)z^p(f_1\star f_2)(z)+\lambda z^{p+1}(f_1\star f_2)'(z)\Big) > \eta_0 \quad (z\in\mathbb{U}),$$

where

$$\eta_0 = 1-4(1-\alpha_1)(1-\alpha_2)\left[1-\frac{1}{2}\,{}_2F_1\left(1,1;\frac{1}{\lambda}+1;\frac{1}{2}\right)\right].$$

The result is the best possible.

Theorem 6. *If $f(z)\in\Sigma_{p,m}$ satisfies the following subordination condition*:

$$(1-\lambda)z^pD^nf(z)+\lambda z^pD^{n+1}f(z)\prec\frac{1+Az}{1+Bz} \quad (z\in\mathbb{U}),$$

then

$$\Re\Big(\big(z^pD^nf(z)\big)^{1/q}\Big) > \rho^{1/q} \quad (q\in\mathbb{N};\ z\in\mathbb{U}),$$

where ρ is given as in Theorem 1. The result is the best possible.

Proof. Defining the function $\phi(z)$ by

$$\phi(z) = z^p D^n f(z) \quad (f \in \Sigma_{p,m};\ z \in \mathbb{U}), \tag{3.26}$$

we see that the function $\phi(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Using the identity (1.5) in (3.26) and differentiating the resulting equation with respect to z, we find that

$$(1-\lambda)z^p D^n f(z) + \lambda z^p D^{n+1} f(z) = \phi(z) + \lambda\, z\phi'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Now, by following the lines of proof of Theorem 1 *mutatis mutandis*, and using the elementary inequality:

$$\Re\big(w^{1/q}\big) \geqq \big(\Re(w)\big)^{1/q} \quad \big(\Re(w) > 0;\ q \in \mathbb{N}\big),$$

we arrive at the result asserted by Theorem 6. □

Upon setting

$$A = \left[{}_2F_1\left(1,1;\frac{1}{\lambda(p+m)}+1;\frac{1}{2}\right)-1\right]\cdot\left[2-{}_2F_1\left(1,1;\frac{1}{\lambda(p+m)}+1;\frac{1}{2}\right)\right]^{-1},$$
$$B=-1,\ n=0, \quad \text{and} \quad q=1$$

in Theorem 6, we deduce Corollary 7 below.

Corollary 7. *If $f(z) \in \Sigma_{p,m}$ satisfies the following inequality*:

$$\Re\big((1+\lambda p)z^p f(z) + \lambda z^{p+1} f'(z)\big) > \frac{3-2\,{}_2F_1\left(1,1;\frac{1}{\lambda(p+m)}+1;\frac{1}{2}\right)}{2\left[2-{}_2F_1\left(1,1;\frac{1}{\lambda(p+m)}+1;\frac{1}{2}\right)\right]} \quad (z \in \mathbb{U}), \tag{3.27}$$

then

$$\Re\big(z^p f(z)\big) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

The result is the best possible.

From Corollary 6 and Theorem 6 (for $m=-p+1$, $A=1-2\eta_0$, $B=-1$, and $q=1$), we deduce the following result.

Corollary 8. *If the functions $f_j(z) \in \Sigma_p$ $(j=1,2)$ satisfy the inequality (3.25), then*

$$\Re\Big(z^p(f_1 \star f_2)(z)\Big) > \eta_0 + (1-\eta_0)\left[{}_2F_1\left(1,1;\frac{1}{\lambda}+1;\frac{1}{2}\right)-1\right] \quad (z \in \mathbb{U}),$$

where η_0 is given as in Corollary 6. The result is the best possible.

Theorem 7. *Let $f(z) \in \Sigma_{p,m}^n(A,B)$ and let $g(z) \in \Sigma_{p,m}$ satisfy the following inequality*:

$$\Re\big(z^p g(z)\big) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Then

$$(f \star g)(z) \in \Sigma_{p,m}^n(A,B).$$

Proof. We have

$$-\frac{z^{p+1}\big(D^n(f\star g)(z)\big)'}{p} = -\frac{z^{p+1}\big(D^n f(z)\big)'}{p} \star z^p g(z) \quad (z \in \mathbb{U}).$$

Since

$$\Re\big(z^p g(z)\big) > \frac{1}{2} \quad (z \in \mathbb{U})$$

and the function

$$\frac{1+Az}{1+Bz}$$

is convex (univalent) in $\mathbb{U}$, it follows from (1.6) and Lemma 5 that

$$(f \star g)(z) \in \Sigma_{p,m}^{n}(A,B).$$

This completes the proof of Theorem 7. □

In view of Corollary 7 and Theorem 7, we have Corollary 9 below.

Corollary 9. *If* $f(z) \in \Sigma_{p,m}^{n}(A,B)$ *and the function* $g(z) \in \Sigma_{p,m}$ *satisfies the inequality* (3.27), *then*

$$(f \star g)(z) \in \Sigma_{p,m}^{n}(A,B).$$

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