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ON THE BOUNDS FOR THE SPECTRAL AND ℓ_p NORMS OF THE KHATRI-RAO PRODUCT OF CAUCHY-HANKEL MATRICES

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ABSTRACT. In this paper we first establish a lower bound and an upper bound for the ℓ_p norms of the Khatri-Rao product of Cauchy-Hankel matrices of the form H_n=$[1/(g+(i+j)h)]_{i,j=1}^n$ for $g=1/2$ and $h=1$ partitioned as

$$H_n = \begin{pmatrix} H_n^{(11)} & H_n^{(12)} \\ H_n^{(21)} & H_n^{(22)} \end{pmatrix}$$

where $H_n^{(ij)}$ is the ijth submatrix of order $m_i \times n_j$ with $H_n^{(11)} = H_{n-1}$. We then present a lower bound and an upper bound for the spectral norm of Khatri-Rao product of these matrices.

Key words and phrases: Cauchy-Hankel matrices, Kronecker product, Khatri-Rao product, Tracy-Singh product, Norm.

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1. INTRODUCTION AND PRELIMINARIES

A Cauchy-Hankel matrix is a matrix that is both a Cauchy matrix (i.e. $(1/(x_i - y_j))_{i,j=1}^n$, $x_i \neq y_j$) and a Hankel matrix (i.e. $(h_{i+j})_{i,j=1}^n$) such that

$$H_n = \left[\frac{1}{g+(i+j)h}\right]_{i,j=1}^n, \tag{1.1}$$

where g and $h \neq 0$ are arbitrary numbers and g/h is not an integer.

Recently, there have been several papers on the norms of Cauchy-Toeplitz matrices and Cauchy-Hankel matrices [2, 3, 12, 21]. Turkmen and Bozkurt [20] have established bounds

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for the spectral norms of the Cauchy-Hankel matrix in the form (1.1) by taking $g = 1/k$ and $h = 1$. Solak and Bozkurt [17] obtained lower and upper bounds for the spectral norm and Euclidean norm of the H_n matrix that has given (1.1). Liu [9] established a connection between the Khatri-Rao and Tracy-Singh products, and present further results including matrix equalities and inequalities involving the two products and also gave two statistical applications. Liu [10] obtained new inequalities involving Khatri-Rao products of positive semidefinite matrices. Neverthless, we know that the Hadamard and Kronecker products play an important role in matrix methods for statistics, see e.g. [18, 11, 8], also these products are studied and applied widely in matrix theory and statistics; see, e.g., [18], [11], [1, 5, 22]. For partitioned matrices the Khatri-Rao product, viewed as a generalized Hadamard product, is discussed and used in [8], [6], [13, 14, 15] and the Tracy-Singh product, as a generalized Kronecker product, is discussed and applied in [7], [19].

The purpose of this paper is to study the bounds for the spectral and the ℓ_p norms of the Khatri-Rao product of two $n \times n$ Cauchy-Hankel matrices of the form (1.1). In this section, we give some preliminaries. In Section 2, we study the spectral norm and the ℓ_p norms of Khatri-Rao product of two $n \times n$ Cauchy-Hankel matrices of the form (1.1) and obtain lower and upper bounds for these norms.

Let A be any $m \times n$ matrix. The ℓ_p norms of the matrix A are defined as

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{\frac{1}{p}} \qquad 1 \leq p < \infty \tag{1.2}$$

and also the spectral norm of matrix A is

$$\|A\|_s = \sqrt{\max_{1 \leq i \leq n} \lambda_i},$$

where the matrix A is $m \times n$ and λ_i are the eigenvalues of $A^H A$ and A^H is a conjugate transpose of matrix A. In the case $p = 2$, the ℓ_2 norm of the matrix A is called its Euclidean norm. The $\|A\|_s$ and $\|A\|_2$ norms are related by the following inequality

$$\frac{1}{\sqrt{n}} \|A\|_2 \leq \|A\|_s . \tag{1.3}$$

The Riemann Zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for complex values of s. While converging only for complex numbers s with $\operatorname{Re} s > 1$, this function can be analytically continued on the whole complex plane (with a single pole at $s = 1$).

The Hurwitz's Zeta function $\zeta(s, a)$ is a generalization of the Riemann's Zeta function $\zeta(s)$ that also known as the generalized Zeta function. It is defined by the formula

$$\zeta(s, a) \equiv \sum_{k=0}^{\infty} \frac{1}{(k + a)^s}$$

for $R[s] > 1$, and by analytic continuation to other $s \neq 1$, where any term with $k + a = 0$ is excluded. For $a > -1$, a globally convergent series for $\zeta(s, a)$ (which, for fixed a, gives an analytic continuation of $\zeta(s, a)$ to the entire complex s - plane except the point $s = 1$) is given by

$$\zeta(s, a) = \frac{1}{s - 1} \sum_{n=0}^{\infty} \frac{1}{n + 1} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} (a + k)^{1-s},$$

see Hasse [4]. The Hurwitz's Zeta function satisfies

$$\begin{aligned}\zeta\left(s,\frac{1}{2}\right) &= \sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)^{-s} \\ &= 2^s\sum_{k=0}^{\infty}(2k+1)^{-s} \\ &= 2^s\left[\zeta(s)-\sum_{k=1}^{\infty}(2k)^{-s}\right] \\ &= 2^s(1-2^{-s})\zeta(s)\end{aligned}$$

$$\zeta\left(s,\frac{1}{2}\right) = (2^s-1)\zeta(s). \tag{1.4}$$

The gamma function can be given by Euler's integral form

$$\Gamma(z) \equiv \int_0^{\infty} t^{z-1}e^{-t}dt.$$

The digamma function is defined as a special function which is given by the logarithmic derivative of the gamma function (or, depending on the definition, the logarithmic derivative of the factorial). Because of this ambiguity, two different notations are sometimes (but not always) used, with

$$\Psi(z) = \frac{d}{dz}\ln\left[\Gamma(z)\right] = \frac{\Gamma'(z)}{\Gamma(z)}$$

defined as the logarithmic derivative of the gamma function $\Gamma(z)$, and

$$F(z) = \frac{d}{dz}\ln(z!)$$

defined as the logarithmic derivative of the factorial function. The nth derivative $\Psi(z)$ is called the polygamma function, denoted $\Psi(n,z)$. The notation $\Psi(n,z)$ is therefore frequently used as the digamma function itself. If $a>0$ and b any number and $n\in\mathbb{Z}^+$ is positive integer, then

$$\lim_{n\to\infty}\Psi(a,n+b) = 0. \tag{1.5}$$

Consider matrices $A=(a_{ij})$ and $C=(c_{ij})$ of order $m\times n$ and $B=(b_{kl})$ of order $p\times q$. Let $A=(A_{ij})$ be partitioned with A_{ij} of order $m_i\times n_j$ as the (i,j)th block submatrix and let $B=(B_{kl})$ be partitioned with B_{kl} of order $p_k\times q_l$ as the (k,l)th block submatrix ($\sum m_i = m, \sum n_j = n, \sum p_k = p$ and $\sum q_l = q$). Four matrix products of A and B, namely the Kronecker, Hadamard, Tracy-Singh and Khatri-Rao products, are defined as follows.

The Kronecker product, also known as tensor product or direct product, is defined to be

$$A\otimes B = (a_{ij}B),$$

where a_{ij} is the ijth scalar element of $A=(a_{ij})$, $a_{ij}B$ is the ijth submatrix of order $p\times q$ and $A\otimes B$ is of order $mp\times nq$.

The Hadamard product, or the Schur product, is defined as

$$A\odot C = (a_{ij}c_{ij}),$$

where a_{ij}, c_{ij} and $a_{ij}c_{ij}$ are the ijth scalar elements of $A=(a_{ij}), C=(c_{ij})$ and $A\odot C$ respectively, and A, C and $A\odot C$ are of order $m\times n$.

The Tracy-Singh product is defined to be

$$A\circ B = (A_{ij}\circ B) \qquad \text{with} \qquad A_{ij}\circ B = (A_{ij}\otimes B_{kl})$$

where A_{ij} is the ijth submatrix of order $m_i \times n_j$, B_{kl} is the klth submatrix of order $p_k \times q_l$, $A_{ij} \otimes B_{kl}$ is the klth submatrix of order $m_ip_k \times n_jq_l$, $A_{ij} \circ B$ is the ijth submatrix of order $m_ip \times n_jq$ and $A \circ B$ is of order $mp \times nq$.

The Khatri-Rao product is defined as

$$A * B = (A_{ij} \otimes B_{ij})$$

where A_{ij} is the ijth submatrix of order $m_i \times n_j, B_{ij}$ is the ijth submatrix of order $p_i \times q_j$, $A_{ij} \otimes B_{ij}$ is the ijth submatrix of order $m_ip_i \times n_jq_j$ and $A * B$ is of order $\left(\sum m_ip_i\right) \times \left(\sum n_jq_j\right)$.

2. THE SPECTRAL AND ℓ_p NORMS OF THE KHATRI-RAO PRODUCT OF TWO $n \times n$ CAUCHY-HANKEL MATRICES

If we substitute $g = 1/2$ and $h = 1$ into the H_n matrix (1.1), then we have

$$H_n = \left[\frac{1}{\frac{1}{2} + (i+j)}\right]_{i,j=1}^{n} \tag{2.1}$$

Theorem 2.1. *Let the matrix $H_n (n \geq 2)$ given in (2.1) be partitioned as*

$$H_n = \begin{pmatrix} H_n^{(11)} & H_n^{(12)} \\ H_n^{(21)} & H_n^{(22)} \end{pmatrix} \tag{2.2}$$

where $H_n^{(ij)}$ is the ijth submatrix of order $m_i \times n_j$ with $H_n^{(11)} = H_{n-1}$. Then

$$\begin{aligned} \|H_n * H_n\|_p^p \leq 2^{2p} \Bigg[2 + \left(\frac{1}{2} - 2^{-p}\right) \zeta(p-1) \\ - \frac{3}{2}\left(1 - 2^{-p}\right)\zeta(p) - \ln 2 \Bigg]^2 + 2^{2p-3}\left[1 - \ln 2\right]^2 + \left(\frac{2}{9}\right)^{2p}. \end{aligned}$$

and

$$\|H_n * H_n\|_p^p \geq 2^{2p-4}\left[\left(\frac{1}{2} - 2^{-p}\right)\zeta(p-1) - \frac{3}{2}\left(1 - 2^{-p}\right)\zeta(p) + 1\right]^2 + 2\left(\frac{2}{7}\right)^{2p}.$$

is valid where $\|\cdot\|_p$ $(3 \leq p < \infty)$ is ℓ_p norm and the operation "$*$" is a Khatri-Rao product.

Proof. Let H_n be defined by (2.1) partitioned as in (2.2). $H_n * H_n$, Khatri-Rao product of two H_n matrices, is obtained as

$$H_n * H_n = \begin{pmatrix} H_n^{(11)} \otimes H_n^{(11)} & H_n^{(12)} \otimes H_n^{(12)} \\ H_n^{(21)} \otimes H_n^{(21)} & H_n^{(22)} \otimes H_n^{(22)} \end{pmatrix}.$$

Using the ℓ_p norm and Khatri-Rao definitions one may easily compute $\|H_n * H_n\|_p$ relative to the above $\left\|H_n^{(ij)} \otimes H_n^{(ij)}\right\|_p$ as shown in (2.3)

$$\|H_n * H_n\|_p^p = \sum_{i,j=1}^{2} \left\|H_n^{(ij)} \otimes H_n^{(ij)}\right\|_p^p \tag{2.3}$$

We may use the equality (1.2) to write

$$\begin{aligned}\left\|H_n^{(11)}\otimes H_n^{(11)}\right\|_p^p &= \left[\sum_{i,j=1}^{n-1}\frac{1}{\left(\frac{1}{2}+i+j\right)^p}\right]^2\\ &= 2^{2p}\left[\sum_{k=1}^{n-1}\frac{k}{(2k+3)^p}+\sum_{k=1}^{n-2}\frac{n-k-1}{(2n+2k+1)^p}\right]^2\\ &= 2^{2p}\left[\left(\sum_{k=2}^{n}\frac{k-1}{(2k+1)^p}+\sum_{k=1}^{n-2}\frac{n-k-1}{(2n+2k+1)^p}\right)\right]^2\\ &= 2^{2p}\left[\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{(2k+1)^{p-1}}-\frac{1}{(2k+1)^p}\right)\right.\\ &\qquad \left.-\sum_{k=1}^{n}\frac{1}{(2k+1)^p}+\sum_{k=1}^{n-2}\frac{n-k-1}{(2n+2k+1)^p}+1\right]^2 \end{aligned}\tag{2.4}$$

From (1.4), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\left(\frac{1}{(2k+1)^{p-1}}-\frac{1}{(2k+1)^p}\right) &= 2^{1-p}\zeta\left(p-1,\frac{1}{2}\right)-2^{-p}\zeta\left(p,\frac{1}{2}\right)\\ &= (1-2^{1-p})\zeta(p-1)-(1-2^{-p})\zeta(p).\end{aligned}\tag{2.5}$$

Also, since

$$\lim_{n\to\infty}\sum_{k=1}^{n-2}\frac{n-k-1}{(2n+2k+1)^p}=\begin{cases}0, & p>2\\ \frac{1}{4}\left(1-\ln 2\right), & p=2\end{cases}\tag{2.6}$$

and from (1.4), (2.4), (2.5), (2.6), we have

$$\left\|H_n^{(11)}\otimes H_n^{(11)}\right\|_p^p\le 2^{2p}\left[\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right)-\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)+2-\ln 2\right]^2.\tag{2.7}$$

Using (2.3) and (2.7) we can write

$$\begin{aligned}\|H_n*H_n\|_p^p &\le 2^{2p}\left[2+\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right)-\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)-\ln 2\right]^2\\ &\qquad +2\left[\sum_{i=1}^{n-1}\frac{1}{\left(\frac{1}{2}+i+n\right)^p}\right]^2+\left[\frac{1}{\left(\frac{1}{2}+2n\right)^p}\right]^2\\ &\le 2^{2p}\left[2+\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right)-\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)-\ln 2\right]^2\\ &\qquad +2^{2p+1}\left[\sum_{i=1}^{n-1}\frac{n-i}{(2n+2i+1)^p}\right]^2+\frac{1}{\left(\frac{1}{2}+2n\right)^{2p}}.\end{aligned}\tag{2.8}$$

Thus, from (2.6) and (2.8) we obtain an upper bound for $\|H_n * H_n\|_p^p$ such that

$$\|H_n * H_n\|_p^p \leq 2^{2p}\left[2+\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right) -\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)-\ln 2\right]^2+2^{2p-3}\left[1-\ln 2\right]^2+\left(\frac{2}{9}\right)^{2p}. \tag{2.9}$$

For the lower bound, if we consider inequality

$$\begin{aligned}\left\|H_n^{(11)}\otimes H_n^{(11)}\right\|_p^p &\geq \left[2^{p-2}\sum_{k=1}^{n-1}\frac{k}{(2k+3)^p}\right]^2 \\ &= \left[2^{p-2}\sum_{k=2}^{n}\frac{k-1}{(2k+1)^p}\right]^2 \\ &= 2^{2p-4}\left[\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right)-\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)+1\right]^2\end{aligned}$$

and equalities (2.3), (2.5), then we have

$$\|H_n * H_n\|_p^p \geq 2^{2p-4}\left[\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right) -\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)+1\right]^2+2\left(\frac{2}{7}\right)^{2p}. \tag{2.10}$$

This is a lower bound for $\|H_n * H_n\|_p^p$. Thus, the proof of the theorem is completed using (2.9) and (2.10). $\square$

Example 2.1. Let

$$\alpha = 2^{2p}\left[2+\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right) -\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)-\ln 2\right]^2+2^{2p-3}\left[1-\ln 2\right]^2+\left(\frac{2}{9}\right)^{2p}$$

$$\beta = 2^{2p-4}\left[\left(\frac{1}{2}-2^{-p}\right)\zeta\left(p-1\right)-\frac{3}{2}\left(1-2^{-p}\right)\zeta\left(p\right)+1\right]^2+2\left(\frac{2}{7}\right)^{2p}$$

and order of $H_n * H_n$ matrix is N. Thus, we have the following values:

N	β	$\|H_n * H_n\|_3$	α
2	0.1932680901	0.1943996774	2.034031369
5	0.1932680901	0.2486967434	2.034031369
10	0.1932680901	0.2949003201	2.034031369
17	0.1932680901	0.3250545239	2.034031369
26	0.1932680901	0.3460881969	2.034031369
37	0.1932680901	0.3615449198	2.034031369
50	0.1932680901	0.3733657155	2.034031369
65	0.1932680901	0.3826914230	2.034031369
81	0.1932680901	0.3902333553	2.034031369

N	β	$\lVert H_n * H_n \rVert_4$	α
2	0.1347849117	0.1654942693	2.294793856
5	0.1347849117	0.2041337591	2.294793856
10	0.1347849117	0.2215153273	2.294793856
17	0.1347849117	0.2305342653	2.294793856
26	0.1347849117	0.2357826204	2.294793856
37	0.1347849117	0.2390987777	2.294793856
50	0.1347849117	0.2413257697	2.294793856
65	0.1347849117	0.2428929188	2.294793856
81	0.1347849117	0.2440372508	2.294793856

N	β	$\lVert H_n * H_n \rVert_5$	α
2	0.1218381759	0.1622386787	2.554705355
5	0.1218381759	0.1845312641	2.554705355
10	0.1218381759	0.1920519007	2.554705355
17	0.1218381759	0.1952045459	2.554705355
26	0.1218381759	0.1967458182	2.554705355
37	0.1218381759	0.1975855071	2.554705355
50	0.1218381759	0.1980810422	2.554705355
65	0.1218381759	0.1983919845	2.554705355
81	0.1218381759	0.1985968085	2.554705355

Now, we will obtain a lower bound and an upper bound for spectral norm of the Khatri-Rao product of two H_n as in (2.1) and partitioned as in (2.2).

To minimize the numerical round-off errors in solving system $Ax = b$, it is normally convenient that the rows of A be properly scaled before the solution procedure begins. One way is to premultiply by the diagonal matrix

$$D = \operatorname{diag}\left\{ \frac{\alpha_1}{r_1(A)}, \frac{\alpha_2}{r_2(A)}, \dots, \frac{\alpha_n}{r_n(A)} \right\}, \tag{2.11}$$

where $r_i(A)$ is the Euclidean norm of the ith row of A and $\alpha_1, \alpha_2, \dots, \alpha_n$ are positive real numbers such that

$$\alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 = n. \tag{2.12}$$

Clearly, the euclidean norm of the coefficient matrix $B = DA$ of the scaled system is equal to $\sqrt{n}$ and if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ then each row of B is a unit vector in the Euclidean norm. Also, we can define $B = AD$,

$$D = \operatorname{diag}\left\{ \frac{\alpha_1}{c_1(A)}, \frac{\alpha_2}{c_2(A)}, \dots, \frac{\alpha_n}{c_n(A)} \right\}, \tag{2.13}$$

where $c_i(A)$ is the Euclidean norm of the ith column of A. Again, $\lVert B \rVert_2 = \sqrt{n}$ and if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 1$ then each column of B is a unit vector in the Euclidean norm.

We now that

$$\lVert B \rVert_2 \leq \lVert D \rVert_2 \cdot \lVert A \rVert_2 \tag{2.14}$$

for B matrix above (see O. Rojo [16]).

Theorem 2.2. *Let the matrix $H_n (n > 2)$ given in (2.1) be partitioned as*

$$H_n = \begin{pmatrix} H_n^{(11)} & H_n^{(12)} \\ H_n^{(21)} & H_n^{(22)} \end{pmatrix}$$

where $H_n^{(ij)}$ is the ijth submatrix of order $m_i \times n_j$ with $H_n^{(11)} = H_{n-1}$ and α_i's $(i = 1, \dots, n)$ be as in (2.12). Then,

$$\|H_n * H_n\|_s \leq \pi^2 + 32 \left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}$$

$$\|H_n * H_n\|_s \geq \left(\sum_{i=1}^{n-1} \frac{\alpha_i^2}{-\Psi(1, n + \frac{1}{2} - i) + \Psi(1, \frac{3}{2} + i)} \right)^{-2} + 32 \left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}$$

is valid where $\|\cdot\|_s$ is spectral norm and the operation "$*$" is a Khatri-Rao product.

Proof. Let H_n be defind by (2.1) and partitioned as in (2.2). $H_n * H_n$, the Khatri-Rao product of two H_n matrices, is obtained as

$$H_n * H_n = \begin{pmatrix} H_n^{(11)} \otimes H_n^{(11)} & H_n^{(12)} \otimes H_n^{(12)} \\ H_n^{(21)} \otimes H_n^{(21)} & H_n^{(22)} \otimes H_n^{(22)} \end{pmatrix}.$$

Using the ℓ_p norm and Khatri-Rao definitions one may easily compute $\|H_n * H_n\|_p$ relative to the above $\left\|H_n^{(ij)} \otimes H_n^{(ij)}\right\|_p$ as shown in (2.3)

$$\|H_n * H_n\|_p^p = \sum_{i,j=1}^{2} \left\|H_n^{(ij)} \otimes H_n^{(ij)}\right\|_p^p$$

First of all, we must establish a function $f(x)$ such that

$$h_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-isx} dx = \frac{1}{\frac{1}{2} + s}, \quad s = 2, 3, \dots, 2n.$$

where h_s are the entries of the matrix H_n . Hence, we must find values of c such that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} c e^{((1/2)+s)ix} e^{-isx} dx = \frac{1}{\frac{1}{2} + s}.$$

Thus, we have

$$\frac{c}{2\pi} \int_{-\pi}^{\pi} e^{(1/2)+s} e^{-isx} dx = \frac{2c}{\pi}$$

and

$$c = \frac{\pi}{2\left(\frac{1}{2} + s\right)}.$$

Hence, we have

$$f(x) = \frac{\pi}{2\left(\frac{1}{2} + s\right)} e^{((1/2)+s)ix}.$$

The function $f(x)$ can be writtten as

$$f(x) = f_1(x) f_2(x),$$

where $f_1(x)$ is a real-valued function and $f_2(x)$ is a function with period 2π and $|f_2(x)| = 1$. Thus, we have

$$f_1(x) = \frac{\pi}{2\left(\frac{1}{2}+s\right)}$$

and

$$f_2(x) = e^{((1/2)+s)ix}.$$

Since $\|H_{n-1}\|_s \leq \sup f_1(x)$,

$$\sum_{k=1}^{n-1} \frac{1}{(2k+5)^2} = -\frac{1}{4}\Psi\left(1, n+\frac{5}{2}\right) + \frac{1}{8}\pi^2 - \frac{259}{225}$$

and from (2.3), we have

$$\|H_n * H_n\|_s \leq \pi^2 + 32\left[-\frac{1}{4}\Psi\left(1, n+\frac{5}{2}\right) + \frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}.$$

Thus, from (1.5) and (2.12) we obtain an upper bound for the spectral norm Khatri-Rao product of two $H_n (n > 2)$ as in (2.1) partitioned as in (2.2) such that

$$\|H_n * H_n\|_s \leq \pi^2 + 32\left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}.$$

Also, we have

$$\|D\|_2 = \left(\sum_{i=1}^{n-1} \frac{\alpha_i^2}{-\Psi(1, n+\frac{1}{2}-i) + \Psi(1, \frac{3}{2}+i)}\right)^{\frac{1}{2}}$$

for D a matrix as defined by (2.13). Since $\|B\|_2 = \sqrt{n-1}$ for B matrix above and from (1.3), we have a lower bound for spectral norm Khatri-Rao product of two $H_n (n > 2)$ as in (2.1) and partitioned as in (2.2) such that

$$\|H_n * H_n\|_s \geq \left(\sum_{i=1}^{n-1} \frac{\alpha_i^2}{-\Psi\left(1, n+\frac{1}{2}-i\right) + \Psi\left(1, \frac{3}{2}+i\right)}\right)^{-2} + 32\left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}.$$

This completes the proof. □

Example 2.2. Let

$$a = \pi^2 + 32\left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561},$$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1,$$

$$\beta = \left(\sum_{i=1}^{n-1} \frac{1}{-\Psi(1, n+\frac{1}{2}-i) + \Psi(1, \frac{3}{2}+i)}\right)^{-2} + 32\left[\frac{1}{8}\pi^2 - \frac{259}{225}\right]^2 + \frac{16}{6561}$$

and order of $H_n * H_n$ matrix is N. We have known that the bounds for $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 1$ are better than those for α_i's $(i = 1, \ldots, n)$ such that $\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = n$. Thus, we have the following values for the spectral norm of $H_n * H_n$:

N	β	$\|H_n * H_n\|_s$	a
5	0.2279281696	0.3909209269	10,09031555
10	0.2234942988	0.5703160868	10,09031555
17	0.2220047678	0.7282096597	10,09031555
26	0.2213922974	0.8664326411	10,09031555
37	0.2211033668	0.9883803285	10,09031555
50	0.2209527402	1.097039615	10,09031555

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