

Journal of Inequalities in Pure and Applied Mathematics

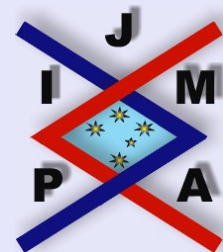
ON AN INEQUALITY OF OSTROWSKI TYPE

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©2000 Victoria University
ISSN (electronic): 1443-5756
225-06



volume 7, issue 4, article 151,
2006.

*Received 22 August, 2006;
accepted 28 August, 2006.*

Communicated by: W.-S. Cheung

Abstract

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Abstract

We prove an inequality of Ostrowski type for p -norm, generalizing a result of Dragomir [1].

2000 Mathematics Subject Classification: 26D15, 26D10.

Key words: Ostrowski's inequality, p -norm.

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1. Introduction

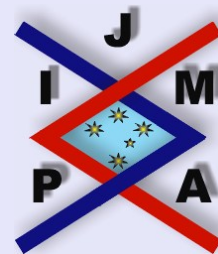
For a differentiable function $f: [a, b] \rightarrow \mathbb{R}$, $a \cdot b > 0$, Dragomir has in [1] proved, using Pompeiu's mean value theorem [3], the following Ostrowski type inequality:

$$(1.1) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq D(x) \cdot \|f - \iota f'\|_\infty,$$

where $\iota(t) = t$, $t \in [a, b]$, and

$$(1.2) \quad D(x) = \frac{b-a}{|x|} \left(\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right).$$

We are going to prove a general estimate with the p -norm, $1 \leq p \leq \infty$, which will for $p = \infty$ give the Dragomir result.



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2. The Main Result

Theorem 2.1. Let the function $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$, and all $x \in [a, b]$, the following inequality holds:

$$(2.1) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq PU(x, p) \cdot \|f - \iota f'\|_p$$

where $\iota(t) = t$, $t \in [a, b]$, and

$$(2.2) \quad PU(x, p) = (b-a)^{\frac{1}{p}-1} \cdot \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right].$$

Note that in cases $(p, q) = (1, \infty)$, $(\infty, 1)$, and $(2, 2)$ the constant $PU(x, p)$ has to be taken as the limit as $p \rightarrow 1, \infty$, and 2 , respectively.

Proof. Define $F: [1/b, 1/a] \rightarrow \mathbb{R}$ by $F(t) := t f(\frac{1}{t})$. The function F is continuous and is differentiable on $(1/b, 1/a)$, and for all $x_1, x_2 \in [1/b, 1/a]$ we have

$$(2.3) \quad \begin{aligned} F(x_1) - F(x_2) &= \int_{x_2}^{x_1} F'(t) dt \\ &= \int_{x_2}^{x_1} f\left(\frac{1}{t}\right) - \frac{1}{t} f'\left(\frac{1}{t}\right) dt \quad \left(\left[u := \frac{1}{t} \right] \right) \end{aligned}$$



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$$(2.4) \quad = - \int_{1/x_2}^{1/x_1} (f(u) - uf'(u)) \frac{1}{u^2} du.$$

Denote $x_1 =: 1/x$ and $x_2 =: 1/t$. Then for all $x, t \in [a, b]$ from (2.4) we get

$$\frac{1}{x} f(x) - \frac{1}{t} f(t) = \int_x^t (f(u) - uf'(u)) \frac{1}{u^2} du,$$

i.e.,

$$(2.5) \quad tf(x) - xf(t) = xt \int_x^t (f(u) - uf'(u)) \frac{1}{u^2} du.$$

Integrating on t and dividing by x , we obtain

$$\frac{b^2 - a^2}{2} \cdot \frac{f(x)}{x} - \int_a^b f(t) dt = \int_a^b t \left(\int_x^t (f(u) - uf'(u)) \frac{1}{u^2} du \right) dt,$$

and therefore

$$(2.6) \quad \left| \frac{b^2 - a^2}{2} \cdot \frac{f(x)}{x} - \int_a^b f(t) dt \right| \\ \leq \int_a^b \left| \int_x^t (f(u) - uf'(u)) \frac{t}{u^2} du \right| dt \\ = \int_a^x \left| \int_t^x (f(u) - uf'(u)) \frac{t}{u^2} du \right| dt \\ + \int_x^b \left| \int_x^t (f(u) - uf'(u)) \frac{t}{u^2} du \right| dt.$$



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First, consider the case $1 < p, q < \infty$. Applying Hölder's inequality, the sum in the last line of (2.5) is

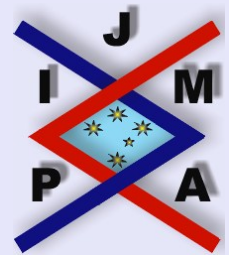
$$\begin{aligned}
 (2.7) \quad &\leq \left(\int_a^x \left(\int_t^x |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
 &+ \left(\int_x^b \left(\int_x^t |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
 &\leq \left(\int_a^b \left(\int_a^b |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} \\
 &\times \left[\left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

The first factor in (2.7) equals

$$(2.8) \quad \left(\int_a^b \left(\int_a^b |f(u) - uf'(u)|^p du \right) dt \right)^{\frac{1}{p}} = (b-a)^{\frac{1}{p}} \|f - \iota f'\|_p,$$

and for the second factor, for $p, q \neq 2$, we get

$$\begin{aligned}
 (2.9) \quad &\left(\int_a^x \left(\int_t^x \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
 &= \left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
 &+ \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}}.
 \end{aligned}$$



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Putting (2.8) and (2.9) into (2.7) and dividing (2.6) and (2.7) by $(b - a)$ gives the required inequality (2.1) in the case $1 < p, q < \infty, p, q \neq 2$.

For $p = q = 2$, instead of (2.9) we obtain

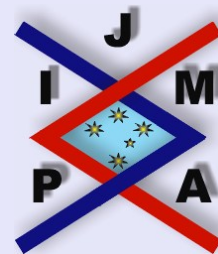
$$(2.10) \quad \left(\int_a^x \left(\int_t^x \frac{t^2}{u^4} du \right) dt \right)^{\frac{1}{2}} + \left(\int_x^b \left(\int_x^t \frac{t^2}{u^4} du \right) dt \right)^{\frac{1}{2}} \\ = \frac{1}{3} \left(\left(\ln \left(\frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left(\ln \left(\frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right)$$

which is easily shown to be equal to the limit of the right hand side in (2.9) for $q \rightarrow 2$, i. e.

$$(2.11) \quad \lim_{p \rightarrow 2} PU(x, p) \\ = \frac{1}{3(b-a)^{\frac{1}{2}}} \left(\left(\ln \left(\frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left(\ln \left(\frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right).$$

Now consider the case $p = \infty, q = 1$. The last line in (2.6) is

$$(2.12) \quad \leq \sup_{a \leq u \leq b} |f(u) - u f'(u)| \\ \times \left(\int_a^x \left(\int_t^x \frac{t}{u^2} du \right) dt + \int_x^b \left(\int_x^t \frac{t}{u^2} du \right) dt \right) \\ = \|f - \iota f'\|_{\infty} \cdot \left(\frac{a^2 + b^2}{2x} + x - a - b \right).$$



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Putting (2.12) into (2.6) and dividing by $(b - a)$ gives

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{1}{b-a} \left(\frac{a^2 + b^2}{2x} + x - a - b \right) \cdot \|f - {}_t f'\|_\infty$$

which reproves the Dragomir's result [1].

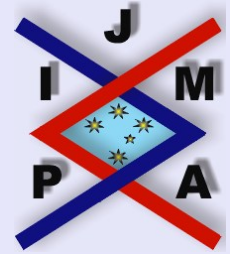
It is easy to see that

$$\frac{1}{b-a} \left(\frac{a^2 + b^2}{2x} + x - a - b \right) = \lim_{p \rightarrow \infty} PU(x, p)$$

proving (2.1) in the case $p = \infty, q = 1$.

Finally consider the case $p = 1, q = \infty$. In this case the last line of (2.6) is

$$\begin{aligned} (2.13) \quad & \leq \int_a^x \left| \int_t^x |f(u) - u f'(u)| \max_{\substack{t \leq u \leq x \\ a \leq t \leq x}} \frac{t}{u^2} du \right| dt \\ & + \int_x^b \left| \int_x^t |f(u) - u f'(u)| \max_{\substack{x \leq u \leq t \\ x \leq t \leq b}} \frac{t}{u^2} du \right| dt \\ & \leq \int_a^b \int_a^b |f(u) - u f'(u)| du dt \cdot \left(\frac{1}{a} + \frac{b}{x^2} \right) \\ & = (b-a) \left(\frac{1}{a} + \frac{b}{x^2} \right) \cdot \|f - {}_t f'\|_1. \end{aligned}$$



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Appending (2.13) to (2.6) and dividing by $(b - a)$ gives

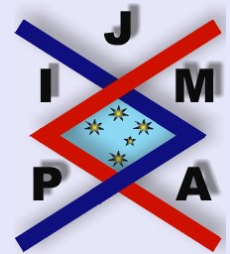
$$(2.14) \quad \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\frac{1}{a} + \frac{b}{x^2} \right) \cdot \|f - {}_t f'\|_1.$$

It is not too difficult to show that

$$(2.15) \quad \lim_{p \rightarrow 1} PU(x, p) = \frac{1}{a} + \frac{b}{x^2},$$

so (2.14) proves formula (2.1) for $p = 1, q = \infty$, proving the theorem. \square

Remark 1. We have considered only the positive case, $0 < a < b$. In the case $a < b < 0$, a similar but more cumbersome formula holds, where most a 's, b 's and x 's have to be replaced by their absolute values.



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3. The Weighted Case

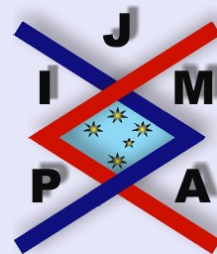
In the weighted case we have the following result:

Theorem 3.1. *Let the function $f: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$, and let $w: [a, b] \rightarrow \mathbb{R}$ be a nonnegative integrable function. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$, and all $x \in [a, b]$, the following inequality holds:*

$$(3.1) \quad \left| \frac{f(x)}{x} \int_a^b t w(t) dt - \int_a^b f(t) w(t) dt \right| \\ \leq \|f - \iota f'\|_p \cdot \frac{(b-a)^{\frac{1}{p}}}{(1-2q)^{\frac{1}{q}}} \cdot \left[\left(x^{1-2q} \int_a^x t^q (w(t))^q dt - \int_a^x t^{1-q} (w(t))^q dt \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_x^b t^{1-q} (w(t))^q dt - x^{1-2q} \int_x^b t^q (w(t))^q dt \right)^{\frac{1}{q}} \right].$$

Proof. Multiplying (2.5) by $w(t)/x$ and integrating on t , we get

$$\frac{f(x)}{x} \int_a^b t w(t) dt - \int_a^b f(t) w(t) dt \\ = \int_a^b t w(t) \left(\int_x^t (f(u) - u f'(u)) \frac{1}{u^2} du \right) dt,$$



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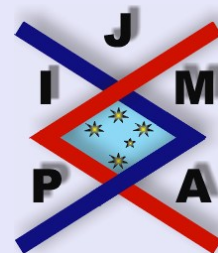
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and as in the proof of Theorem 2.1 we have

$$\begin{aligned}
 & \left| \frac{f(x)}{x} \int_a^b t w(t) dt - \int_a^b f(t) w(t) dt \right| \\
 & \leq \int_a^x \left| \int_t^x (f(u) - u f'(u)) \frac{t w(t)}{u^2} du \right| dt \\
 & \quad + \int_x^b \left| \int_x^t (f(u) - u f'(u)) \frac{t w(t)}{u^2} du \right| dt \\
 & \leq \left(\int_a^x \left(\int_t^x |f(u) - u f'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_a^x \left(\int_t^x \frac{t^q (w(t))^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
 & \quad + \left(\int_x^b \left(\int_x^t |f(u) - u f'(u)|^p du \right) dt \right)^{\frac{1}{p}} \cdot \left(\int_x^b \left(\int_x^t \frac{t^q (w(t))^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \\
 & \leq \left(\int_a^b \left(\int_a^b |f(u) - u f'(u)|^p du \right) dt \right)^{\frac{1}{p}} \\
 & \quad \cdot \left[\left(\int_a^x \left(\int_t^x \frac{t^q (w(t))^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q (w(t))^q}{u^{2q}} du \right) dt \right)^{\frac{1}{q}} \right]
 \end{aligned}$$

which gives (3.1). □



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