

A NEW REFINEMENT OF THE HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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Abstract: In this paper we establish a new refinement of the Hermite-Hadamard inequality for convex functions.



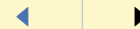
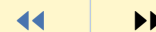
Hermite-Hadamard Inequality

G. Zabandan

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1. Introduction

Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function, then the following inequality:

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality [5].

In recent years there have been many extensions, generalizations and similar results of the inequality (1.1).

In [2], Dragomir established the following theorem which is a refinement of the left side of (1.1).

Theorem 1.1. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and H is defined on $[0, 1]$ by*

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx,$$

then H is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) = H(0) \leq H(t) \leq H(1) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In [6] Yang and Hong established the following theorem which is a refinement of the right side of inequality (1.1).

Theorem 1.2. *If $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, and F is defined by*

$$F(t) = \frac{1}{2(b-a)} \int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right) + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right) \right] dx,$$

then F is convex, increasing on $[0, 1]$, and for all $t \in [0, 1]$, we have

$$\frac{1}{b-a} \int_a^b f(x) dx = F(0) \leq F(t) \leq F(1) = \frac{f(a) + f(b)}{2}.$$

In this paper we establish a refinement of the both sides of inequality (1.1). For this we first define two sequences $\{x_n\}$ and $\{y_n\}$ by

$$\begin{aligned} (1.2) \quad x_n &= \frac{1}{2^n} \sum_{i=1}^{2^n} f \left(a + i \frac{b-a}{2^n} - \frac{b-a}{2^{n+1}} \right) \\ &= \frac{1}{2^n} \sum_{i=1}^{2^n} f \left(a + \left(i - \frac{1}{2} \right) \frac{b-a}{2^n} \right), \end{aligned}$$

$$\begin{aligned} (1.3) \quad y_n &= \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} \left[f \left(\left(1 - \frac{i}{2^n} \right) a + \frac{i}{2^n} b \right) + f \left(\left(1 - \frac{i-1}{2^n} \right) a + \frac{i-1}{2^n} b \right) \right] \\ &= \frac{1}{2^{n+1}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f \left(\left(1 - \frac{i}{2^n} \right) a + \frac{i}{2^n} b \right) \right] \end{aligned}$$

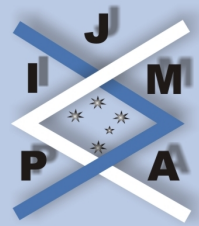
and we prove the following

$$\begin{aligned} f \left(\frac{a+b}{2} \right) &= x_0 \leq \frac{1}{2} \left[f \left(\frac{3a+b}{4} \right) + f \left(\frac{a+3b}{4} \right) \right] \\ &= x_1 \leq \dots \leq x_n \leq \dots \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq \dots \leq y_n \leq \dots \leq y_1 \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \\
&\leq y_0 = \frac{f(a) + f(b)}{2},
\end{aligned}$$

which is a new refinement of the Hermite-Hadamard inequality (1.1). For a similar discussion, see [1] or the monograph online [7, p. 19 – 22].



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2. A Refinement Result

In this section, using the terminologies of the introduction, we refine the Hermite-Hadamard inequality via the sequences $\{x_n\}$ and $\{y_n\}$.

Theorem 2.1. *Let f be a convex function on $[a, b]$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq x_n \leq \frac{1}{b-a} \int_a^b f(x) dx \leq y_n \leq \frac{f(a) + f(b)}{2}.$$

Proof. By the right side of Hermite-Hadamard inequality (1.1) we have

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{b-a} \sum_{i=1}^{2^n} \int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}} f(x) dx \\ &\leq \frac{1}{b-a} \sum_{i=1}^{2^n} \left(a + i\frac{b-a}{2^n} - a - (i-1)\frac{b-a}{2^n} \right) \frac{f\left(a + i\frac{b-a}{2^n}\right) + f\left(a + (i-1)\frac{b-a}{2^n}\right)}{2} \\ &= \frac{1}{2^{n+1}} \left[\sum_{i=1}^{2^n} f\left[\left(1 - \frac{i}{2^n}\right)a + \frac{i}{2^n}b\right] + f\left[\left(1 - \frac{i-1}{2^n}\right)a + \frac{i-1}{2^n}b\right] \right] \\ &= y_n. \end{aligned}$$

By the convexity of f we obtain

$$y_n \leq \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} \left[\left(1 - \frac{i}{2^n}\right) f(a) + \frac{i}{2^n} f(b) + \left(1 - \frac{i-1}{2^n}\right) f(a) + \frac{i-1}{2^n} f(b) \right]$$



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$$\begin{aligned}
 &= \frac{1}{2^{n+1}} \left[f(a) \sum_{i=1}^{2^n} \left(2 - \frac{i}{2^{n-1}} + \frac{1}{2^n} \right) + f(b) \sum_{i=1}^{2^n} \left(\frac{i}{2^{n-1}} - \frac{1}{2^n} \right) \right] \\
 &= \frac{1}{2^{n+1}} \left[f(a) \left(2^{n+1} - \frac{1}{2^{n-1}} \frac{2^n(2^n+1)}{2} + \frac{2^n}{2^n} \right) + f(b) \left(\frac{1}{2^{n-1}} \cdot \frac{2^n(2^n+1)}{2} - \frac{2^n}{2^n} \right) \right] \\
 &= \frac{1}{2^{n+1}} [f(a)(2^{n+1} - 2^n) + f(b)(2^n)] = \frac{f(a) + f(b)}{2},
 \end{aligned}$$

so

$$\frac{1}{b-a} \int_a^b f(x) dx \leq y_n \leq \frac{f(a) + f(b)}{2}.$$

On the other hand, by the left side of inequality (1.1) we have

$$\begin{aligned}
 \frac{1}{b-a} \int_a^b f(x) dx &= \frac{1}{b-a} \sum_{i=1}^{2^n} \int_{a+(i-1)\frac{b-a}{2^n}}^{a+i\frac{b-a}{2^n}} f(x) dx \geq \frac{1}{b-a} \sum_{i=1}^{2^n} \frac{b-a}{2^n}, \\
 f \left(\frac{a + i\frac{b-a}{2^n} + a + (i-1)\frac{b-a}{2^n}}{2} \right) &= \frac{1}{2^n} \sum_{i=1}^{2^n} f \left(a + i\frac{b-a}{2^n} - \frac{b-a}{2^{n+1}} \right) = x_n.
 \end{aligned}$$

By the convexity of f and Jensen's inequality we obtain

$$\begin{aligned}
 x_n &= \frac{1}{2^n} \sum_{i=1}^{2^n} f \left(a + i\frac{b-a}{2^n} - \frac{b-a}{2^{n+1}} \right) \\
 &\geq f \left[\frac{1}{2^n} \sum_{i=1}^{2^n} \left(a + i\frac{b-a}{2^n} - \frac{b-a}{2^{n+1}} \right) \right] \\
 &= f \left[\frac{1}{2^n} \left(2^n a + \frac{b-a}{2^n} \cdot \frac{2^n(2^n+1)}{2} - \frac{b-a}{2^{n+1}} 2^n \right) \right]
 \end{aligned}$$

$$= f\left(a + \frac{b-a}{2}\right) = f\left(\frac{a+b}{2}\right).$$

□

Theorem 2.2. Let f be a convex function on $[a, b]$, then $\{x_n\}$ is increasing, $\{y_n\}$ is decreasing and

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \frac{1}{b-a} \int_a^b f(x) dx.$$

Proof. We have

$$\begin{aligned} x_n &= \frac{1}{2^n} \sum_{i=1}^{2^n} f\left(a + i \frac{b-a}{2^n} - \frac{b-a}{2^{n+1}}\right) \\ &= \frac{1}{2^n} \sum_{i=1}^{2^n} f\left(\frac{(2^{n+1} - 2i + 1)a + (2i - 1)b}{2^{n+1}}\right) \\ &= \frac{1}{2^n} \sum_{i=1}^{2^n} f\left(\frac{1}{2} \cdot \frac{(2^{n+3} - 8i + 4)a + (8i - 4)b}{2^{n+2}}\right) \\ &= \frac{1}{2^n} \sum_{i=1}^{2^n} f\left(\frac{1}{2} \cdot \frac{(2^{n+2} + 3 - 4i)a + (4i - 3)b + (2^{n+2} + 1 - 4i)a + (4i - 1)b}{2^{n+2}}\right) \\ &\leq \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} f\left(\frac{(2^{n+2} + 3 - 4i)a + (4i - 3)b}{2^{n+2}}\right) \\ &\quad + \frac{1}{2^{n+1}} \sum_{i=1}^{2^n} f\left(\frac{(2^{n+2} + 1 - 4i)a + (4i - 1)b}{2^{n+2}}\right) \end{aligned}$$



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set $A = \{1, 3, \dots, 2^{n+1} - 1\}$ and $B = \{2, 4, \dots, 2^{n+1}\}$, thus we obtain

$$\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2} + 3 - 4i)a + (4i - 3)b}{2^{n+2}}\right) = \sum_A f\left(\frac{(2^{n+2} + 1 - 2i)a + (2i - 1)b}{2^{n+2}}\right)$$

$$\sum_{i=1}^{2^n} f\left(\frac{(2^{n+2} + 1 - 4i)a + (4i - 1)b}{2^{n+2}}\right) = \sum_B f\left(\frac{(2^{n+2} + 1 - 2i)a + (2i - 1)b}{2^{n+2}}\right),$$

which implies that

$$x_n \leq \frac{1}{2^{n+1}} \left[\sum_{A \cup B} f\left(\frac{(2^{n+2} + 1 - 2i)a + (2i - 1)b}{2^{n+2}}\right) \right] = x_{n+1},$$

so $\{x_n\}$ is increasing. On the other hand we have

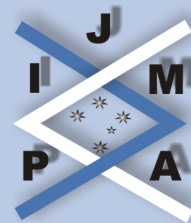
$$y_{n+1} = \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^{n+1}-1} f\left[\left(1 - \frac{i}{2^{n+1}}\right)a + \frac{i}{2^{n+1}}b\right] \right]$$

$$= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^{n+1}-1} f\left(\frac{(2^{n+1} - i)a + ib}{2^{n+1}}\right) \right].$$

Setting $C = \{2, 4, 6, \dots, 2^{n+1} - 2\}$, we obtain

$$y_{n+1} = \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i \in C} f\left(\frac{(2^{n+1} - i)a + ib}{2^{n+1}}\right) + 2 \sum_{i \in A} f\left(\frac{(2^{n+1} - i)a + ib}{2^{n+1}}\right) \right]$$

$$= \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f\left(\frac{(2^{n+1} - 2i)a + 2ib}{2^{n+1}}\right) \right]$$



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$$\begin{aligned}
 & + 2 \sum_{i=1}^{2^n} f \left(\frac{(2^{n+1} - 2i + 1)a + (2i - 1)b}{2^{n+1}} \right) \Big] \\
 = & \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
 & \left. + 2 \sum_{i=1}^{2^n} f \left(\frac{1}{2} \cdot \frac{(2^n - i)a + ib + (2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
 \leq & \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
 & \left. + \sum_{i=1}^{2^n} f \left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n} f \left(\frac{(2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
 = & \frac{1}{2^{n+2}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) \right. \\
 & \left. + f(b) + f(a) + \sum_{i=2}^{2^n} f \left(\frac{(2^n - i + 1)a + (i - 1)b}{2^n} \right) \right] \\
 = & \frac{1}{2^{n+2}} \left[2f(a) + 2f(b) + 3 \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) + \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) \right] \\
 = & \frac{1}{2^{n+1}} \left[f(a) + f(b) + 2 \sum_{i=1}^{2^n-1} f \left(\frac{(2^n - i)a + ib}{2^n} \right) \right] = y_n,
 \end{aligned}$$

so $\{y_n\}$ is decreasing.



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For the proof of the last assertions, since f is continuous on $[a, b]$, we use the following well known equality:

$$\lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{i=1}^n f\left(a + i \frac{b-a}{n}\right) = \int_a^b f(x) dx.$$

So we obtain

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \frac{1}{b-a} \int_a^b f(x) dx.$$

□

Remark 1. Let f be a convex function on $[a, b]$. In conclusion, we can state that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) = x_0 &\leq \frac{1}{2} f\left[\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &= x_1 \leq \dots \leq x_n \leq \dots \\ &\leq \frac{1}{b-a} \int_a^b f(x) dx \leq \dots \leq y_n \leq \dots \leq y_1 \\ &= \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] \\ &\leq y_0 = \frac{f(a) + f(b)}{2}. \end{aligned}$$

3. Applications for Special Means

Recall the following means

a) The arithmetic mean

$$A(a, b) = \frac{a + b}{2} \quad (a, b > 0);$$

b) The geometric mean

$$G(a, b) = \sqrt{ab} \quad (a, b > 0);$$

c) The harmonic mean

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} \quad (a, b > 0);$$

d) The logarithmic mean

$$L(a, b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & b \neq a; \\ a & b = 0; \end{cases} \quad (a, b > 0).$$

We define the two new means by the following:

e) The n -harmonic mean

$$H_n(a, b) = 2^{n+1} \left[\frac{1}{a} + 2 \sum_{i=1}^{2^n-1} \frac{1}{\left(1 - \frac{i}{2^n}\right)a + \frac{i}{2^n}b} + \frac{1}{b} \right]^{-1}$$

$(n = 0, 1, 2, \dots, \quad a, b > 0)$



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f) The n -arithmetic mean

$$A_n(a, b) = 2^n \left[\sum_{i=1}^{2^n} \frac{1}{\left(1 - \frac{i}{2^n} + \frac{1}{2^{n+1}}\right) a + \left(\frac{i}{2^n} - \frac{1}{2^{n+1}}\right) b} \right]^{-1}$$

$(n = 0, 1, 2, \dots; a, b > 0).$

It is clear that $H_0(a, b) = H(a, b)$ and $A_0(a, b) = A(a, b)$. By the above terminology we have the following simple proposition:

Proposition 3.1. *Let $0 < a < b < \infty$. Then we have*

$$H(a, b) \leq H_n(a, b) \leq L(a, b) \leq A_n(a, b) \leq A(a, b),$$
$$\lim_{n \rightarrow \infty} H_n(a, b) = \lim_{n \rightarrow \infty} A_n(a, b) = L(a, b).$$

Proof. Let $f : [a, b] \rightarrow (0, \infty)$, $f(x) = \frac{1}{x}$ and use Remark 1. We omit the details. □

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