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CONVEX FUNCTIONS IN A HALF-PLANE, II

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Abstract

In the present paper we obtain new sufficient conditions for the univalence and convexity of an analytic function defined in the upper half-plane. In particular, in the case of hydrodynamically normalized functions, we obtain by a different method a known result concerning the convexity and univalence of an analytic function defined in a half-plane.

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I dedicate this paper to the memory of my dear father, Professor Nicolae N. Pascu.

Contents

1. Introduction

In the present paper, we continue the work in [4], by obtaining new sufficient conditions for the convexity and univalence for analytic functions defined in the upper half-plane (Theorems 2.2, 2.3 and 2.4). In particular, under the additional hypothesis (1.2) below, they become necessary and sufficient conditions for convexity and univalence in a half-plane (Corollary 2.5), obtaining thus by a different method the results in [5] and [6].

We begin by establishing the notation and with some preliminary results needed for the proofs.

We denote by $D = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ the upper half-plane in $\mathbb{C}$ and for $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ we let T_ε be the angular domain defined by:

$$T_\varepsilon = \left\{ z \in \mathbb{C}^* : \frac{\pi}{2} - \varepsilon < \arg(z) < \frac{\pi}{2} + \varepsilon \right\}. \tag{1.1}$$

We say that a function $f : D \to \mathbb{C}$ is *convex* if f is univalent in D and $f(D)$ is a convex domain.

For an arbitrarily chosen positive real number $y_0 > 0$ we denote by $\mathcal{A}_{y_0}$ the class of functions $f : D \to \mathbb{C}$ analytic in the upper half-plane D satisfying $f(iy_0) = 0$ and such that $f'(z) \neq 0$ for any $z \in D$. In particular, for $y_0 = 1$ we will denote $\mathcal{A}_1 = \mathcal{A}$.

We will refer to the following normalization condition for analytic functions $f : D \to \mathbb{C}$ as the *hydrodynamic normalization*:

$$\lim_{z \to \infty, z \in D} (f(z) - z) = ai, \tag{1.2}$$

where $a \geq 0$ is a non-negative real number, and we will denote by $\mathcal{H}_1$ the class of analytic functions $f : D \to \mathbb{C}$ satisfying this condition in the particular case $a = 0$.

For analytic functions satisfying the above normalization condition, J. Stankiewicz and Z. Stankiewicz obtained (see [5] and [6]) the following necessary and sufficient condition for convexity and univalence in a half-plane:

Theorem 1.1. *If the function $f \in \mathcal{H}_1$ satisfies:*

$$f'(z) \neq 0, \quad \textit{for all } z \in D \tag{1.3}$$

and

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0, \quad \textit{for all } z \in D, \tag{1.4}$$

then f is a convex function.

In order to prove our main result we need the following results from [2]:

Lemma 1.2. *If the function $f : D \to D$ is analytic in D, then for any $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ the following limits exist and we have the equalities:*

$$\lim_{z\to\infty, z\in T_\varepsilon} \frac{f(z)}{z} = \lim_{z\to\infty, z\in T_\varepsilon} f'(z) = c,$$

where $c \geq 0$ is a non-negative real number.

Moreover, for any $z \in D$ we have the inequality

$$\operatorname{Im} f(z) \geq c \operatorname{Im} z, \tag{1.5}$$

and if there exists $z_0 \in D$ such that we have equality in the inequality (1.5), then there exists a real number a such that

$$f(z) = cz + a, \quad \textit{for all } z \in D.$$

Lemma 1.3. *If the function $f : D \to D$ is analytic in D and hydrodynamically normalized, then for any $\varepsilon \in (0, \frac{\pi}{2})$ and any natural number $n \geq 2$ we have*

$$\lim_{z\to\infty, z\in T_\varepsilon} \left(z^n f^{(n)}(z)\right) = 0.$$

2. Main Results

Let us consider the family of domains $D_{r,s}$ in the complex plane, defined by

$$D_{r,s} = \{z \in \mathbb{C} : |z| < r, \ \operatorname{Im} z > s\},$$

where r and s are positive real numbers, $0 < s < r$ (see Figure 1).

Let us note that for any $r > 1$ and $0 < s < 1$ we have the inclusion $D_{r,s} \subset D$, and that for any $z \in D$ arbitrarily fixed, there exists $r_z > 0$ and $s_z > 0$ such that $z \in D_{r,s}$ for any $r > r_z$ and any $0 < s < s_z$ (for example, we can choose r_z and s_z such that they satisfy the conditions $r_z > |z|$ and $s_z \in (0, \operatorname{Im} z)$).

We denote by $\Gamma_{r,s} = c_r \cup d_s$ the boundary of the domain $D_{r,s}$, where c_r and d_s are the arc of the circle, respectively the line segment, defined by:

$$\begin{cases} c_r = \{z \in \mathbb{C} : |z| = r, \ \ z \geq s\} \\ d_s = \{z \in \mathbb{C} : |z| \leq r, \ z = s\} \end{cases}.$$

The curve $\Gamma_{r,s}$ has an exterior normal vector at any point, except for the points a and b (with $\arg a < \arg b$) where the line segment d_s and the arc of the circle c_r meet (see Figure 1). The exterior normal vector to the curve $f(c_r)$ at the point $f(z)$, with $z = re^{it} \in c_r$, $t \in (\arg a, \arg b)$, has the argument

$$\varphi(t) = \arg(zf'(z)), \tag{2.1}$$

and the exterior normal vector to the curve $f(d_s)$ at the point $f(z)$, with $z = x + is \in d_s$, $x \in (\operatorname{Re} b, \operatorname{Re} a)$, has the argument

$$\psi(x) = -\frac{\pi}{2} + \arg f'(x + is). \tag{2.2}$$

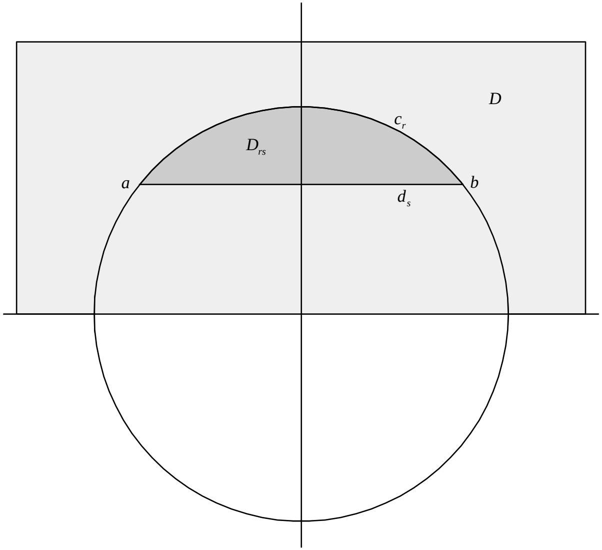

Figure 1: The domain D_{rs}.

Definition 2.1. *We say that the function $f \in \mathcal{A}$ is convex on the curve $\Gamma_{r,s}$ if the argument of the exterior normal vector to the curve $f(\Gamma_{r,s}) - \{f(a), f(b)\}$ is an increasing function.*

Remark 1. *In particular, the condition in the above theorem is satisfied if the functions φ and ψ defined by (2.1)–(2.2) are increasing functions.*

Let us note that for $z = re^{it} \in c_r$, we have:

$$\frac{\partial}{\partial t} \log \left(re^{it} f'\left(re^{it}\right)\right) = i\left(\frac{re^{it} f''\left(re^{it}\right)}{f'\left(re^{it}\right)} + 1\right) = \frac{\partial}{\partial t} \ln \left|re^{it} f'\left(re^{it}\right)\right| + i\varphi'(t),$$

and for $z \in d_s$:

$$\frac{\partial}{\partial x} \log f'(x+is) = \frac{f''(x+is)}{f'(x+is)} = \frac{\partial}{\partial x} \ln |f'(x+is)| + i\psi'(x+is).$$

We obtain therefore

$$\varphi'(t) = \frac{re^{it} f''\left(re^{it}\right)}{f'\left(re^{it}\right)} + 1,$$

for $re^{it} \in c_r$, and

$$\psi'(x+is) = \frac{f''(x+is)}{f'(x+is)},$$

for $x + is \in d_s$, and from the previous observation it follows that if the function $f \in \mathcal{A}$ satisfies the inequalities

$$\left\{ \begin{array}{ll} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in c_r \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in d_s \end{array} \right. , \tag{2.3}$$

the function f is convex on the curve $\Gamma_{r,s}$, and therefore $f(D_{r,s})$ is a convex domain.

Since the function f has in the domain $D_{r,s}$ bounded by the curve $\Gamma_{r,s}$ a simple zero, from the argument principle it follows that the total variation of the argument of the function f on the curve $\Gamma_{r,s}$ is 2π, and therefore f is injective on the curve $\Gamma_{r,s}$. From the principle of univalence on the boundary, it follows that the function f is univalent $D_{r,s}$.

We obtained the following:

Theorem 2.1. *If the function f belongs to the class $\mathcal{A}$ and there exist real numbers $0 < s < 1 < r$ such that conditions (2.3) are satisfied, then the function f is univalent in the domain $D_{r,s}$ and $f(D_{r,s})$ is a convex domain.*

More generally, we have the following:

Theorem 2.2. *If the function $f : D \to \mathbb{C}$ belongs to the class $\mathcal{A}$ and there exist real numbers $0 < s_0 < 1 < r_0$ such that:*

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0 \tag{2.4}$$

for any $z \in D$ with $|z| > r_0$, and

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0 \tag{2.5}$$

for any $z \in D$ with $\operatorname{Im} z < s_0$, then the function f is convex and univalent in the half-plane D.

Proof. Let z_1 and z_2 be arbitrarily fixed distinct points in the half-plane D. For any $r > r^* = \max\{|z_1|, |z_2|\}$ and any $s \in (0, s^*)$, where $s^* = \min\{\operatorname{Im} z_1, \operatorname{Im} z_2\}$, the points z_1 and z_2 belong to the domain $D_{r,s}$.

From the hypothesis (2.4) and (2.5) and using the Remark 1 it follows that for any $r > r_0$ and $s \in (0, s_0)$ the function f is univalent in the domain $D_{r,s}$, and that $f(D_{r,s})$ is a convex domain.

Therefore, choosing $r > \max\{r_0, r^*\}$ and $s \in (0, s_1)$, where $s_1 = \min\{s_0,\ s^*\}$, it follows that the points z_1 and z_2 belong to the domain $D_{r,s}$, and since the function f is univalent in the domain $D_{r,s}$, we obtain that $f(z_1) \neq f(z_2)$.

Since z_1 and z_2 were arbitrarily chosen in the half-plane D, it follows that the function f is univalent in D, concluding the first part of the proof.

In order to show that $f(D)$ is a convex domain, we consider w_1 and w_2 arbitrarily fixed distinct points in $f(D)$, and let $z_1 = f^{-1}(w_1)$ and $z_2 = f^{-1}(w_2)$ be their preimages.

Repeating the above proof it follows that the points z_1 and z_2 belong to the domain $D_{r,s}$ (for any $r > \max\{r_0, r^*\}$ and $s \in (0, s_1)$, where $s_1 = \min\{s_0,\ s^*\}$, in the notation above), and therefore we obtain that $w_1 = f(z_1) \in f(D_{r,s})$ and $w_2 = f(z_2) \in f(D_{r,s})$.

Since $f(D_{r,s})$ is a convex domain, it follows that the line segment $[w_1, w_2]$ is also contained in the domain $f(D_{r,s})$, and since $f(D_{r,s}) \subset f(D)$, we obtain that $[w_1, w_2] \subset f(D)$.

Since $w_1, w_2 \in f(D)$ were arbitrarily chosen, it follows that $f(D)$ is a convex domain, concluding the proof. □

Remark 2. *The point $z_0 = i$, in which the functions f belonging to the class $\mathcal{A} = \mathcal{A}_1$ are normalized can be replaced by any point $z_0 = iy_0$, with $y_0 >$*

0. *Repeating the proof of the previous theorem with this new choice for the normalization condition, we obtain the following result which generalizes the previous theorem:*

Theorem 2.3. *If the function $f : D \to \mathbb{C}$ belongs to the class $\mathcal{A}_{y_0}$ for some $y_0 > 0$, and there exist real numbers $0 < s_0 < y_0 < r_0$ such that*

$$\begin{cases} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in D,\ |z| > r_0 \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in D,\ z \in (0, s_0) \end{cases},$$

then the function f is univalent and convex in the half-plane D.

Remark 3. *By noticing that the function $f : D \to \mathbb{C}$ is convex and univalent in D if and only the function $\widetilde{f} : D \to \mathbb{C}$, $\widetilde{f}(z) = f(z) - f(iy_0)$ is convex and univalent in D, for any arbitrarily chosen point $y_0 > 0$, and replacing the function f in the previous theorem by $\widetilde{f}(z) = f(z) - f(iy_0)$, we can eliminate from the hypothesis of this theorem the condition $f(iy_0) = 0$, obtaining the following more general result:*

Theorem 2.4. *If the function $f : D \to \mathbb{C}$ is analytic in D, satisfies $f'(z) \neq 0$ for all $z \in D$ and there exist real numbers $0 < s_0 < r_0$ such that the following inequalities hold:*

$$\begin{cases} \dfrac{zf''(z)}{f'(z)} + 1 > 0, & z \in D,\ |z| > r_0 \\ \dfrac{f''(z)}{f'(z)} > 0, & z \in D,\ z \in (0, s_0) \end{cases}, \tag{2.6}$$

then the function f is convex and univalent in the half-plane D.

Example 2.1. *For $a \in \mathbb{R}$, consider the function $f_a : D \to \mathbb{C}$ defined by*

$$f_a(z) = z^a, \quad z \in D,$$

where we have chosen the determination of the power function corresponding to the principal branch of the logarithm, that is:

$$z^a = e^{a \log z}, \quad z \in D,$$

where $\log z$ denotes the principal branch of the logarithm (with $\log i = i\frac{\pi}{2}$).

We have

$$\begin{aligned} f_a'(i) &= a i^{a-1} \\ &= a\left(\cos\frac{(a-1)\pi}{2} + i \sin\frac{(a-1)\pi}{2}\right) \\ &\neq 0, \end{aligned}$$

for any $a \neq 0$.

For an arbitrarily chosen $z \in D$ we have:

$$\begin{aligned} \frac{f_a''(z)}{f_a'(z)} &= (a-1)\frac{1}{z} \\ &= -\frac{(a-1)\overline{z}}{|z|^2} \\ &> 0 \end{aligned}$$

for any $a < 1$*, and also*

$$\frac{zf_a''(z)}{f_a'(z)} + 1 = (a-1) + 1$$
$$= a$$
$$> 0$$

for any $a > 0$.

It follows that the hypotheses of the previous theorem are satisfied for any $a \in (0,1)$*, and according to this theorem it follows that the function* $f_a(z) = z^a$ *(*$z \in D$*) is convex and univalent in the half-plane* D *for any* $a \in (0,1)$*.*

It is easy to see that the function $f_a(z) = z^a$*,* $z \in D$*, is convex and univalent for any* $a \in (-1,0) \cup (0,1)$*, and therefore the previous theorem gives only sufficient conditions for the convexity and univalence of an analytic function defined in the upper half-plane* D*.*

Remark 4. *As shown in [4], the condition*

$$\frac{f''(z)}{f'(z)} > 0, \quad z \in D,$$

is a necessary condition (but not also a sufficient one) for an analytic function in D *to be convex and univalent in* D*.*

However, in the case of a hydrodynamically normalized function, as shown in Theorem 1.1 (see [5] and [6]), this becomes also a sufficient condition for the convexity and the univalence in the half-plane D*. We recall that the hydrodynamic normalization used by Stankiewicz in is given by*

$$\lim_{z\to\infty, z\in D} (f(z) - z) = 0. \tag{2.7}$$

In particular, in the case of analytic and hydrodynamically normalized functions in the upper half-plane, from Theorem 2.4 we can obtain as a consequence a new proof of the last cited result, namely a necessary and sufficient condition for the convexity and the univalence of an analytic, hydrodynamically normalized function defined in the half-plane, as follows:

Corollary 2.5. *If the function $f : D \to \mathbb{C}$ is analytic and hydrodynamically normalized by (1.2) in the half-plane D, and it satisfies*

$$f'(z) \neq 0 \quad \text{for all } z \in D \tag{2.8}$$

and

$$\operatorname{Im} \frac{f''(z)}{f'(z)} > 0, \quad \text{for all } z \in D, \tag{2.9}$$

then the function f is convex and univalent in the half-plane D.

Proof. Since f satisfies the hydrodynamic normalization condition

$$\lim_{z\to\infty, z\in D} (f(z) - z - ai) = 0,$$

for some $a \geq 0$, it follows that for any $\varepsilon' > 0$ there exists $r > 0$ such that for $z \in D$ with $|z| > r$ we have:

$$|\operatorname{Im}(f(z) - z - ai)| \leq |f(z) - z - ai| < \varepsilon',$$

and therefore we obtain

$$\operatorname{Im} f(z) > \operatorname{Im} z + a - \varepsilon',$$

for any $z \in D$ with $|z| > r$.

Choosing $y_0 = \max\{r, \varepsilon - a\}$ and considering the auxiliary function $g : D \to \mathbb{C}$ defined by

$$g(z) = f(z + 2iy_0)$$

it follows that for all $z \in D$ we have:

$$\begin{aligned} g(z) &= f(z + 2iy_0) \\ &> z + 2y_0 + a - \varepsilon \\ &> y_0 > 0, \end{aligned}$$

which shows that $g : D \to D$.

Since the function f is hydrodynamically normalized, the function g is also hydrodynamically normalized, and from Lemma 1.2 we obtain

$$\begin{aligned} \lim_{z\to\infty, z\in T_\varepsilon} f'(z + 2iy_0) &= \lim_{z\to\infty, z\in T_\varepsilon} g'(z) \\ &= \lim_{z\to\infty, z\in T_\varepsilon} \frac{g(z)}{z} = 1, \end{aligned}$$

since from the hydrodynamic normalization condition we have

$$\begin{aligned} \lim_{z\to\infty, z\in D} \frac{g(z)}{z} - 1 &= \lim_{z\to\infty, z\in D} \frac{g(z) - z}{z} \\ &= \frac{\lim_{z\to\infty, z\in D} g(z) - z}{\lim_{z\to\infty, z\in D} z} \\ &= \frac{ai}{\lim_{z\to\infty, z\in D} z} = 0, \end{aligned}$$

and therefore we obtain $\lim\limits_{z\to\infty, z\in T_\varepsilon} \frac{g(z)}{z} = 1$, for any $\varepsilon \in (0, \frac{\pi}{2})$.

From Lemma 1.3, applied to the function g in the particular case $n = 2$, we obtain:

$$\lim_{z\to\infty, z\in T_\varepsilon} \left[z^2 g''(z)\right] = 0,$$

for any $\varepsilon \in (0, \frac{\pi}{2})$, and therefore we obtain

$$\lim_{z\to\infty, z\in T_\varepsilon} \left[z^2 f''(z)\right] = \lim_{z\to\infty, z\in T_\varepsilon} \left[(z - 2iy_0)^2 g''(z - 2iy_0) \frac{z^2}{(z - 2iy_0)^2}\right] = 0.$$

Since $\lim\limits_{z\to\infty, z\in D} f'(z) = 1$, we obtain

$$\lim_{z\to\infty, z\in T_\varepsilon} \frac{z f''(z)}{f'(z)} = 0,$$

for any $\varepsilon \in (0, \frac{\pi}{2})$.

It follows that for any $\varepsilon \in (0, \frac{\pi}{2})$ arbitrarily fixed, there exists $r_0 > 0$ such that

$$\frac{z f''(z)}{f'(z)} + 1 > 0,$$

for any $z \in T_\varepsilon$ with $|z| > r_0$.

Following the proof Theorem 2.4 it can be seen that this inequality together with the hypotheses (2.8) and (2.9) suffices for the proof, and therefore the function f is convex and univalent in the half-plane D, concluding the proof. □

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