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A PROOF OF TWO CONJECTURES RELATED TO THE ERDÖS-DEBRUNNER INEQUALITY

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ABSTRACT. In this paper we prove some results which imply two conjectures proposed by Janous on an extension to the p-th power-mean of the Erdös–Debrunner inequality relating the areas of the four sub-triangles formed by connecting three arbitrary points on the sides of a given triangle.

Key words and phrases: Erdös-Debrunner inequality, Areas, Extrema, p-th power mean.

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1. MOTIVATION

Given a triangle ABC and three arbitrary points on the sides AB, AC, BC, the Erdös-Debrunner inequality [1] states that

$$F_0 \geq \min(F_1, F_2, F_3), \tag{1.1}$$

where F_0 is the area of the middle formed triangle DEF and F_1, F_2, F_3 are the areas of the surrounding triangles (see Figure 1.1).

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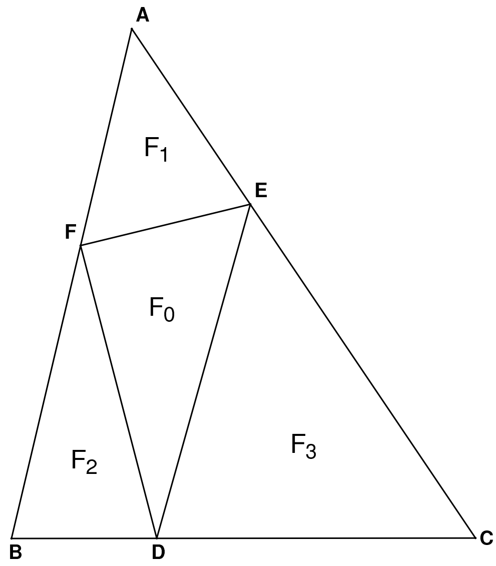

Figure 1.1: Triangle $\triangle ABC$

The p-th power-mean is defined for p on the extended real line by

$$M_p(x_1, x_2, \ldots, x_n) = \begin{cases} \min(x_1, \ldots, x_n), & \text{if } p = -\infty, \\ \left(\frac{\sum_{i=1}^n x_i^p}{n}\right)^{\frac{1}{p}}, & \text{if } p \neq 0, \\ M_0 = \sqrt[n]{\prod_{i=1}^n x_i}, & \text{if } p = 0, \\ \max(x_1, \ldots, x_n), & \text{if } p = \infty. \end{cases}$$

It is known (see [2, Chapter 3]) that M_p is a nondecreasing function of p. Thus, it is natural to ask whether (1.1) can be improved to:

$$F_0 \geq M_p(F_1, F_2, F_3). \tag{1.2}$$

The author of [4] investigated the maximum value of p, denoted here by p_{max}, for which (1.2) is true, showing that $-1 \leq p_{max} \leq -(\frac{\ln 3}{\ln 2} - 1)$ (and disproving a previously published claim).

Since $p_{max} < 0$, by setting $x = \frac{BD}{AE}\frac{AC}{BC}$, $y = \frac{EC}{FB}\frac{AB}{AC}$, $z = \frac{AF}{DC}\frac{BC}{AB}$, and $q = -p$, it is shown in [4] that (1.2) is equivalent to

$$f(x, y, z) := g(x, y)^q + g(y, z)^q + g(z, x)^q \geq 3, \tag{1.3}$$

where $g(x, y) := \frac{1}{x} + y - 1$, q_{min}, the analogue of p_{max}, satisfies $\frac{\ln 3}{\ln 2} - 1 \leq q_{min} \leq 1$, and the variables are such that $g(x, y) \geq 0$, $g(y, z) \geq 0$, $g(z, x) \geq 0$ and $x, y, z > 0$.

Let us introduce the natural domain of f, say $\mathcal{D}$, to be the set of all triples $(x, y, z) \in \mathbb{R}^3$ with $x, y, z > 0$ and $g(x, y) \geq 0$, $g(y, z) \geq 0$ and $g(z, x) \geq 0$. Since $f(x, y, z) \geq 0$, the function f has an infimum on $\mathcal{D}$. Let us denote this infimum by m.

To complete the analysis begun in [4], the author proposed the following two conjectures.

Conjecture 1.1. *For any $q \geq q_0 = \frac{\ln 3}{\ln 2} - 1$, if $f(x, y, z) = m$, then $xyz = 1$.*

Conjecture 1.2. *If $q \geq q_0$, then $m = 3$.*

In this paper we prove (Theorem 2.1) that for every $q > 0$, the function f has a minimum m, and if this infimum is attained for $(x, y, z) \in \mathcal{D}$, then $xyz = 1$. Moreover, we show (Theorem 3.1) that for every $q > 0$ we have $m = \min\{3, 2^{q+1}\}$. Our results are more general than Conjectures 1.1 and 1.2 above, and imply them. After the initial submission of our paper, we learned that the initial conjectures of Janous were also proved by Mascioni [5]. However, our methods are different and Mascioni's Theorem can be obtained from our Theorem 3.12 for $q = q_0$. In other words, we extend the Erdös-Debrunner inequality to the range $p < 0$, and $p = -q_0 = -\frac{\ln(3/2)}{\ln(2)}$ is just a particular value of p for which $C_p = 1$ in Theorem 3.12. This range can be extended for $p > 0$ only in the trivial way, i.e., $F_0 \geq 0 \cdot M_p(F_1, F_2, F_3)$, since $F_0 = 0$ and $M_p(F_1, F_2, F_3) \neq 0$ if, for instance, the point F coincides with the point B and the point E coincides with the point C. As shown next, because the minimum of f is attained at the same point for every $p > -q_0$, we cannot have an inequality of the type $F_0 \geq C\sqrt[3]{F_1F_2F_3}$ with $C > 0$ either. So, our Theorem 3.12 is, in a sense, just as far as one can go along these lines in generalizing the Erdös-Debrunner inequality. Of course, one may try to show that there is a constant $c_p > 0$ such that

$$F_0 \leq c_p M_p(F_1, F_2, F_3), \qquad p \geq 0,$$

however, that is beyond the scope of this paper.

2. PROOF OF CONJECTURE 1.1

We are going to prove the following more general theorem from which Conjecture 1.1 follows.

Theorem 2.1. *For every $q > 0$, the function f defined by (1.3) has a minimum m and if $f(x, y, z) = m$ for some $(x, y, z) \in \mathcal{D}$ then $xyz = 1$.*

Proof. Since $f(1, 1, 1) = 3$ and $f(2, 1/2, 1) = 2^{q+1}$ we see that

$$0 \leq m \leq \min\{3, 2^{q+1}\}.$$

Since $g(x, y) > y - 1$, we see that if $y > 1 + 3^{\frac{1}{q}} =: a$ then $f(x, y, z) > 3$. Similarly, $f(x, y, z) > 3$ if x or z is greater than a. On the other hand, if $x < \frac{1}{a}$ then $g(x, y) > 1/x - 1 > a - 1 = 3^{1/q}$, which implies that $f(x, y, z) > 3$, again. Clearly, if y or z are less than $1/a$ we also have $f(x, y, z) > 3$. Hence, we can introduce the compact domain

$$\mathcal{C} := \left\{ (x, y, z) \middle| \frac{1}{a} \leq x, y, z \leq a,\ g(x, y) \geq 0,\ g(y, z) \geq 0 \text{ and } g(z, x) \geq 0 \right\},$$

which has the property that

$$m = \inf\{f(x, y, z) | (x, y, z) \in \mathcal{C}\}. \tag{2.1}$$

Since any continuous function defined on a compact set attains its infimum, we infer that m is a minimum for f. Moreover, every point at which f takes the value m must be in $\mathcal{C}$.

Let us assume now that we have such a point (x, y, z) as in the statement of Theorem 2.1: $f(x, y, z) = m$. We will consider first the case in which (x, y, z) is in the interior of $\mathcal{C}$.

By the first derivative test (sometimes called Fermat's principle) for local extrema, this point must be a critical point. So, $\frac{\partial f(x,y,z)}{\partial x} = 0$, which is equivalent to

$$x^2 = \frac{g(x, y)^{q-1}}{g(z, x)^{q-1}}.$$

Hence the system

$$\nabla f(x, y, z) = (0, 0, 0) \tag{2.2}$$

is equivalent to

$$x^2 = \frac{g(x,y)^{q-1}}{g(z,x)^{q-1}}; \quad y^2 = \frac{g(y,z)^{q-1}}{g(x,y)^{q-1}}; \quad z^2 = \frac{g(z,x)^{q-1}}{g(y,z)^{q-1}}. \tag{2.3}$$

Multiplying the equalities in (2.3) gives $xyz = 1$, and this proves the theorem when the infimum occurs at an interior point of $\mathcal{C}$.

Now let us assume that the minimum of f is attained at a point (x, y, z) on the boundary of $\mathcal{C}$. Clearly the boundary of $\mathcal{C}$ is

$$\{(x,y,z) \in \mathcal{C} | \{x,y,z\} \cap \{a, 1/a\} \neq \emptyset \quad \text{or} \quad g(x,y)g(y,z)g(z,x) = 0\}.$$

We distinguish several cases.

Case 1: First, if $x = a$, since $1/z > 0$, we have

$$f(x,y,z) \geq \left(\frac{1}{z} + x - 1\right)^q > (a-1)^q = 3 \geq m.$$

Thus, we cannot have $f(x, y, z) = m$ in this situation. Similarly, we exclude the possibility that y or z is equal to a.

Case 2: If $x = 1/a$, because $y > 0$, it follows that

$$f(x,y,z) \geq \left(\frac{1}{x} + y - 1\right)^q > (a-1)^q = 3 \geq m.$$

Again, this implies that $f(x, y, z) = m$ is not possible. Likewise, we can exclude the cases in which y, or z is $1/a$.

Case 3: Let us consider now the case in which $g(x, y) = 0$, that is $y = \frac{x-1}{x}$ (observe that we need $x > 1$). Therefore, $f(x, y, z) = f(x, \frac{x-1}{x}, z)$ becomes the following function of two variables

$$\begin{aligned} k(x,z) &= \left(\frac{x}{x-1} + z - 1\right)^q + \left(\frac{1}{z} + x - 1\right)^q \\ &= \left(z + \frac{1}{x-1}\right)^q + \left(\frac{1}{z} + x - 1\right)^q. \end{aligned}$$

Hence, using the arithmetic-geometric inequality, we obtain

$$\begin{aligned} \left(z + \frac{1}{x-1}\right)^q + \left(\frac{1}{z} + x - 1\right)^q &\geq 2\sqrt{\left(\frac{1}{x-1} + z\right)^q \left(\frac{1}{z} + x - 1\right)^q} \\ &= 2\sqrt{\left[2 + z(x-1) + \frac{1}{z(x-1)}\right]^q} \\ &\geq 2^{q+1}, \end{aligned} \tag{2.4}$$

where we have used $X + 1/X \geq 2$ (for $X > 0$). We observe that if $m = 2^{q+1}$ (this is equivalent to $q \leq q_0$), since $f(x, y, z) = m$, we must have equality in (2.4), which, in particular, implies that $z = \frac{1}{x-1}$, that is, $xyz = 1$. If $m < 2^{q+1}$, then (2.4) shows that we cannot have $f(x, y, z) = m$. Either way, the conjecture is also true in this situation. The other cases are treated in a similar way. □

3. RESULTS IMPLYING CONJECTURE 1.2

We are going to prove a result slightly more general than Conjecture 1.2:

Theorem 3.1. *Assume the notations of Section 2. Then, for every* $q > 0$ *we have* $m = \min\{3, 2^{q+1}\}$.

In [4], Theorem 3.1 was shown to be true for $\frac{\ln 3}{\ln 2} - 1 \leq q \leq 1$. So we are going to assume without loss of generality that $q < 1$ throughout. Based on what we have shown in Section 2, we can let $z = \frac{1}{xy}$ and study the minimum of the function $h(x, y) = f(x, y, \frac{1}{xy})$ on the trace of the domain $\mathcal{C}$ in the space of the first two variables:

$$\mathcal{H} = \left\{ (x, y)|\ x, y \in [1/a, a] \text{ and } \frac{x+1}{x} \geq y \geq \frac{|x-1|}{x} \right\}.$$

Before we continue with the analysis of the critical points inside the domain $\mathcal{H}$ we want to expedite the boundary analysis. We define $A := 1/x + y - 1$, $B := 1/y + 1/(xy) - 1$ and $C := xy + x - 1$. It is a simple matter to show

$$ABC + AB + AC + BC = 4. \tag{3.1}$$

If (x, y) is on the boundary of $\mathcal{H}$, then either $y = \frac{x+1}{x}$, or $y = \frac{|x-1|}{x}$. The first possibility is equivalent to $B = 0$, and the second is equivalent to $A = 0$ (if $x > 1$), or $C = 0$ (if $x < 1$). Now, if $C = 0$ then $AB = 4$. Hence

$$f(x, y, z) \geq A^q + B^q + C^q = A^q + B^q \geq 2\sqrt{(AB)^q} = 2^{1+q}.$$

Similar arguments can be used for the cases $A = 0$ or $B = 0$. Hence, since $h(1, 2) = 2^{q+1}$ we obtain the following result.

Lemma 3.2. *The minimum of* h *on the boundary of* $\mathcal{H}$*, say* $\partial\mathcal{H}$*, is*

$$\min\{h(x, y)|(x, y) \in \partial\,\mathcal{H}\} = 2^{q+1}. \tag{3.2}$$

Next, we analyze critical points inside $\mathcal{H}$. By Fermat's principle, these critical points will satisfy $\frac{\partial h}{\partial x} = 0$, $\frac{\partial h}{\partial y} = 0$, that is,

$$-\frac{1}{x^2} qA^{q-1} - \frac{1}{x^2 y} qB^{q-1} + (y+1)qC^{q-1} = 0,$$

and

$$qA^{q-1} - \frac{x+1}{xy^2} qB^{q-1} + xqC^{q-1} = 0.$$

We remove the common factor q in both of these equations to obtain

$$-\frac{1}{x^2} A^{q-1} - \frac{1}{x^2 y} B^{q-1} + (y+1)C^{q-1} = 0, \tag{3.3}$$

$$A^{q-1} - \frac{x+1}{xy^2} B^{q-1} + xC^{q-1} = 0. \tag{3.4}$$

Solving for A^{q-1} in (3.4) and substituting in (3.3) we get

$$-\frac{x+1}{x^3 y^2} B^{q-1} + \frac{1}{x} C^{q-1} - \frac{1}{x^2 y} B^{q-1} + (y+1)C^{q-1} = 0$$

or

$$\frac{xy + x + 1}{x} C^{q-1} = \frac{x + 1 + xy}{x^3 y^2} B^{q-1}.$$

Since $xy+x+1>0$, $x>0$, by simplifying the previous equation we obtain

$$C^{q-1}=\frac{B^{q-1}}{x^2y^2}. \tag{3.5}$$

Moreover, replacing (3.5) in (3.4), say, we get

$$A^{q-1}-\frac{x+1}{xy^2}B^{q-1}+x\frac{B^{q-1}}{x^2y^2}=0,$$

which implies

$$\frac{A^{q-1}}{x^2}=\frac{B^{q-1}}{x^2y^2}. \tag{3.6}$$

Therefore, if we put (3.5) and (3.6) together, we obtain

$$\frac{A^{q-1}}{x^2}=\frac{B^{q-1}}{x^2y^2}=C^{q-1}. \tag{3.7}$$

The equality $\frac{A^{q-1}}{x^2}=C^{q-1}$ is equivalent to

$$x^{\frac{2}{1-q}}\left(\frac{1}{x}+y-1\right)=xy+x-1.$$

If we introduce the new variable $s=\frac{1+q}{1-q}>1$, the last equality can be written as $yx(1-x^s)=(x^s+1)(1-x)$.

Similarly, the equality $\frac{A^{q-1}}{x^2}=\frac{B^{q-1}}{x^2y^2}$ can be manipulated in the same way to obtain

$$\frac{1}{x}+y-1=y^{\frac{2}{1-q}}\left(\frac{1}{y}+\frac{1}{xy}-1\right), \qquad \text{or}$$
$$\frac{1}{x}(1-y^s)=(1-y)(1+y^s).$$

So, the two equations in (3.7) give the critical points (inside the domain $\mathcal{H}$), which can be classified in the the following way:

- (C_1): (1,1);
- (C_2): $\{(x,1) : x\neq 1$ satisfies $x(1-x^s)=(x^s+1)(1-x)\}$;
- (C_3): $\{(1,y) : y\neq 1$ satisfies $(1-y^s)=(1-y)(1+y^s)\}$;
- (C_4): $\left\{(x,y) : y=\frac{(x^s+1)(x-1)}{x(x^s-1)} \text{ and } x=\frac{y^s-1}{(y-1)(y^s+1)},\ x\neq 1, y\neq 1\right\}$.

Let

$$\phi(t)=\begin{cases}\frac{(t^s+1)(t-1)}{t(t^s-1)} & \text{if } 1\neq t>0,\\ \frac{2}{s} & \text{if } t=1,\end{cases}$$

which is continuous for all $t>0$. Since it is going to be useful later, we note that ϕ satisfies

$$\phi\left(\frac{1}{t}\right)=t\phi(t), \qquad \text{for all } t>0. \tag{3.8}$$

Thus (C_2) is the set of all $(x,1)$ ($x\neq 1$) with $\phi(x)=1$; (C_3) is the set of all $(1,y)$ ($y\neq 1$) with $\phi(1/y)=1$; and (C_4) is the set of all (x,y) ($x\neq 1, y\neq 1$) with

$$\begin{cases} y=\phi(x)\\ x=\frac{1}{\phi(1/y)}.\end{cases} \tag{3.9}$$

Remark 3.3. Due to (3.8), the class (C_3) is in fact the set of all points $(1, y)$, where $y = 1/x$ and $(x, 1)$ is in (C_2).

To determine the nature of the critical points, we compute the second partial derivatives, and analyze the Hessian of h at these critical points. Using relations (3.7) we obtain:

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2} &= q(q-1)\left(\frac{1}{x^4}A^{q-2} + \frac{1}{x^4y^2}B^{q-2} + (y+1)^2C^{q-2}\right)\\ &\qquad + q\left(\frac{2}{x^3}A^{q-1} + \frac{2}{x^3y}B^{q-1}\right)\\ &= \frac{2q(1+y)C^{q-1}}{x} - q(1-q)C^{q-1}\left(\frac{1}{x^2A} + \frac{1}{x^2B} + \frac{(y+1)^2}{C}\right)\\ &= \frac{qC^{q-1}}{x}\left(2(1+y) - (1-q)\frac{(A+B)C + x^2(y+1)^2AB}{xABC}\right)\\ &= \frac{q(q+1)}{x^2ABC^{2-q}}\left(ABC(C+1) - \frac{4}{s}\right),\end{aligned} \tag{3.10}$$

using the fact that $x(y+1) = C+1$.

Similarly, we get

$$\begin{aligned}\frac{\partial^2 h}{\partial y^2} &= q(q-1)\left(A^{q-2} + \frac{(x+1)^2}{x^2y^4}B^{q-2} + x^2C^{q-2}\right) + q\frac{2(x+1)}{xy^3}B^{q-1}\\ &= \frac{2qx(1+x)C^{q-1}}{y} - q(1-q)C^{q-1}\left(\frac{x^2}{A} + \frac{(x+1)^2}{y^2B} + \frac{x^2}{C}\right)\\ &= \frac{qC^{q-1}}{y}\left(2x(1+x) - (1-q)\frac{x^2y^2(A+C)B + (x+1)^2AC}{yABC}\right)\\ &= \frac{q(q+1)x^2}{ABC^{2-q}}\left(ABC(B+1) - \frac{4}{s}\right),\end{aligned} \tag{3.11}$$

using $xy(B+1) = x+1$.

Further, the mixed second derivative is

$$\begin{aligned}\frac{\partial^2 h}{\partial x \partial y} &= q(q-1)\left(-\frac{1}{x^2}A^{q-2} + \frac{x+1}{x^3y^3}B^{q-2} + x(y+1)C^{q-2}\right)\\ &\qquad + q\left(\frac{1}{x^2y^2}B^{q-1} + C^{q-1}\right)\\ &= 2qC^{q-1} - q(1-q)C^{q-1}\left(-\frac{1}{A} + \frac{x+1}{xyB} + \frac{x(y+1)}{C}\right)\\ &= qC^{q-1}\left(1 + q - (1-q)\frac{AC(B+1) + AB - BC}{ABC}\right)\\ &= \frac{q(q+1)}{ABC^{2-q}}\left(ABC - \frac{2}{s}(2 - BC)\right),\end{aligned} \tag{3.12}$$

using the identities $xy(B+1) = x+1$, and $x(y+1) = C+1$.

The discriminant (determinant of the Hessian)

$$D := \frac{\partial^2 h}{\partial x^2} \cdot \frac{\partial^2 h}{\partial y^2} - \left(\frac{\partial^2 h}{\partial x \partial y}\right)^2$$

can be calculated using (3.10), (3.11) and (3.12) to obtain

$$\begin{aligned} D &= \frac{q^2(q+1)^2}{A^2B^2C^{4-2q}}\Big(A^2B^2C^2\left((B+1)(C+1)-1\right) \\ &\qquad - \frac{4}{s}ABC\left(B+C+2-(2-BC)\right) + \frac{4}{s^2}\left(4-(2-BC)^2\right)\Big) \\ &= \frac{q^2(q+1)^2}{A^2B^2C^{4-2q}}\Big(A^2B^2C^2(BC+B+C) \\ &\qquad - \frac{4}{s}ABC(BC+B+C) + \frac{4}{s^2}(4BC-B^2C^2)\Big). \end{aligned}$$

Now, by (3.1) we have

$$\begin{aligned} 4BC - B^2C^2 &= BC(4-BC) \\ &= BC(ABC+AB+AC) \\ &= ABC(BC+B+C) \end{aligned}$$

and so we have the factor $ABC(BC+B+C)$ in all the terms above. This implies that the discriminant of h (at the critical points, that is, assuming relations (3.7)) can be simplified to

$$D = \frac{q^2(q+1)^2}{ABC^{3-2q}}(BC+B+C)\left(ABC+\frac{4}{s^2}-\frac{4}{s}\right). \tag{3.13}$$

Our next lemma classifies the critical point $(1,1)$.

Lemma 3.4. *For $q \geq 1/3$, the point $(1,1)$ is a local minimum. For $q < 1/3$ the critical point $(1,1)$ is not a point of local minimum.*

Proof. If $q = 1/3$, $h(1,1) = 3$, so, since $h(x,y) = f(x,y,\frac{1}{xy}) \geq 3$ by inequality (1.3), we establish that $(1,1)$ is a local minimum point of h. Assume $q \neq 1/3$. For $x = 1$ and $y = 1$ the formulae established above become

$$\begin{aligned} \frac{\partial^2 h}{\partial x^2}(1,1) = \frac{\partial^2 h}{\partial y^2}(1,1) &= 2q(3q-1) > 0, \\ \frac{\partial^2 h}{\partial x \partial y}(1,1) &= q(3q-1) > 0 \end{aligned}$$

and

$$D = 3q^2(3q-1)^2.$$

Hence, the Hessian is positive definite and so we have a local minimum at this point (cf. [3, Theorem 2.9.7, p. 74]). For the second part, observe that $D(1,1) > 0$, but $\frac{\partial^2 h}{\partial x^2}(1,1) < 0$ if $q < 1/3$, and so $(1,1)$ is not a local minimum if $q < 1/3$. □

Theorem 3.5. *If $q \neq 1/3$, there exists only one solution x_0 of $\phi(x) = 1$, $0 < x \neq 1$, such that*

(a) $x_0 \in \left(\frac{1}{2}, \frac{s}{2(s-1)}\right)$ *if* $q > 1/3$ $(s > 2)$;

(b) $x_0 \in \left[2^{\frac{1}{s-1}} - \frac{s^{\frac{s}{s-1}} - s}{2(s-1)}, 2^{\frac{1}{s-1}} - \frac{1}{2(s-1)}\right)$ *if* $q < 1/3$ $(1 < s < 2)$.

Furthermore, there is only one solution $y_0 = 1/x_0$ to $\phi(1/y) = 1$, $0 < y \neq 1$. If $q = 1/3$ $(s = 2)$, there are no positive solutions for $\phi(x) = 1$, $0 < x \neq 1$, or $\phi(1/y) = 1$, $0 < y \neq 1$.

Proof. First, assume $q = 1/3$. Then $s = 2$. It is straightforward to show that $(x,1)$ is in (C_2) implies $x = 1$. However, $x = 1$ is not allowed. Similarly, $(1,y)$ is in (C_3) implies $y = 0$, or 1, which are not allowed. Thus, if $q = 1/3$, there are no positive solutions for $\phi(x) = 1$, $0 < x \neq 1$, or $\phi(1/y) = 1$, $0 < y \neq 1$.

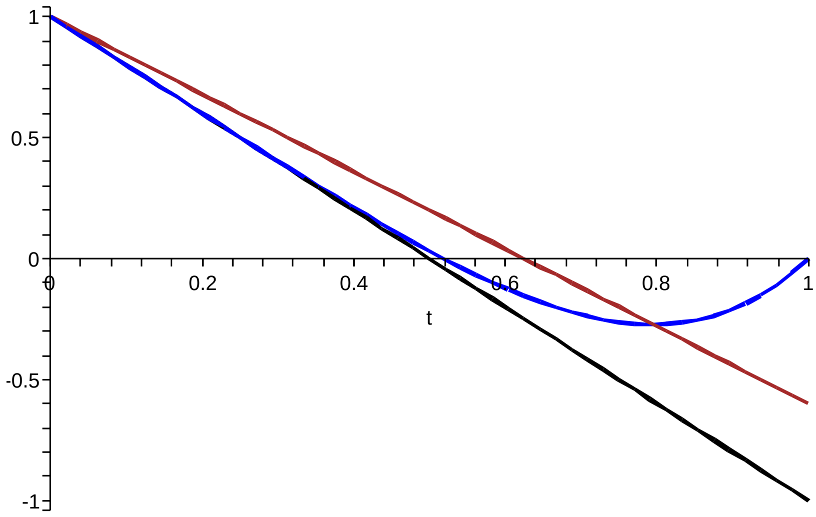

Figure 3.1: The graph of ψ

Now we shall assume throughout that $q \neq 1/3$. Let us observe that the equation $\phi(x) = 1$ can be written equivalently as $\psi(x) = 0$ ($x \neq 1$), where

$$\psi(t) := t^s - 2t + 1, \quad t \geq 0.$$

We first assume that $q > 1/3$, which is equivalent to $s > 2$. The derivative of ψ is $\psi'^{s-1} - 2$ which has only one critical point $t_0 = (2/s)^{\frac{1}{s-1}}$. Since $s > 2$, we obtain that $t_0 < 1$. We have $\psi(0) = 1$, $\psi(1) = 0$ and then automatically

$$\psi(t_0) = (2/s)^{\frac{s}{s-1}} - 2(2/s)^{\frac{s}{s-1}} + 1 = 1 - (s-1)(2/s)^{\frac{s}{s-1}} < 0.$$

The second derivative of ψ is: ψ''^{s-2}. This shows that ψ is a convex function and so its graph lies above any of its tangent lines and below any secant line passing through its graph, as in Figure 3.1.

We conclude that x_0 is between the intersection of the tangent line at $(0,1)$ with the x-axis and the intersection between the secant line connecting $(0,1)$ and $(t_0, \psi(t_0))$ with the x-axis.

Since $\psi'(0) = -2$, the equation of the tangent line is $y - 1 = -2x$ and so its intersection with the x-axis is $(1/2, 0)$. The equation of the secant line through $(0,1)$ and $(t_0, \psi(t_0))$ is $y - 1 = \frac{1-\psi(t_0)}{-t_0}x$, or $y = 1 - \frac{(s-1)2}{s}x$. This gives the intersection with the x-axis: $(\frac{s}{2(s-1)}, 0)$. Therefore the first part of our theorem is proved. The last claim is shown similarly. $\square$

Remark 3.6. As q approaches 1 from below, s becomes large and the interval around x_0 (part (a) in Theorem 3.5) is very small.

Theorem 3.7. *The critical points in (C_2) and (C_3) are not points of local minimum for h.*

Proof. We show that the Hessian of h is not positive semi-definite by showing that the discriminant D is less than zero.

We will treat only the critical points of type (C_2), since the case (C_3) is similar. We get $A = A(x_0, 1) = 1/x_0$, $B = B(x_0, 1) = 1/x_0$ and $C = C(x_0, 1) = 2x_0 - 1$.

The condition $D < 0$ is the same as

$$\frac{2x_0 - 1}{x_0^2} + \frac{4}{s^2} - \frac{4}{s} < 0,$$

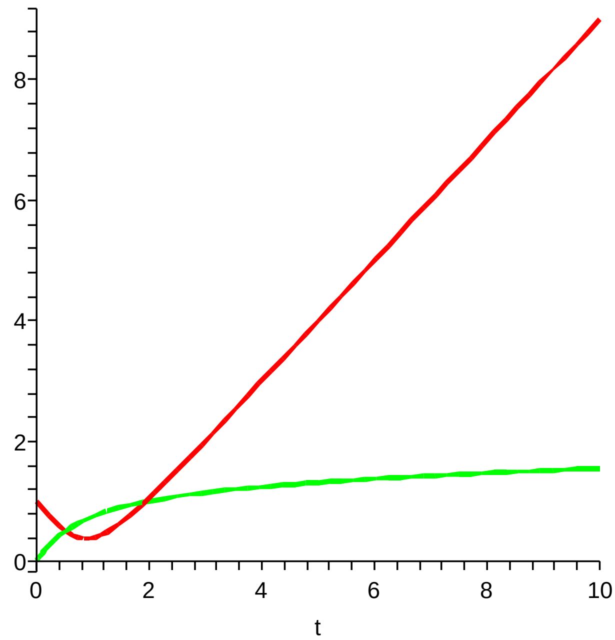

Figure 3.2: The graphs of γ_1, γ_2

which is equivalent to

$$s^2(2x_0 - 1) - 4x_0^2(s-1) = (s - 2x_0)(2(s-1)x_0 - s) < 0$$

or

$$x_0 \in \left(-\infty, \tfrac{s}{2}\right) \cup \left(\tfrac{s}{2(s-1)}, \infty\right), \quad \text{if } q \le 1/3 \ (1 < s \le 2) \ \text{ and}$$
(3.14)
$$x_0 \in \left(-\infty, \tfrac{s}{2(s-1)}\right) \cup \left(\tfrac{s}{2}, \infty\right) \quad \text{if } q > 1/3 \ (s > 2).$$

By Theorem 3.5 parts (a) and (b), and the inequality $2^{\frac{1}{s-1}} - \frac{s^{\frac{s}{s-1}} - s}{2(s-1)} > \frac{s}{2(s-1)}$ that can be easily checked, we see that $D < 0$, which completes the proof. □

Next, we define *the* two functions

$$\gamma_1(t) := \frac{(t-1)(1+t^s)}{t^s - 1} \quad \text{and} \quad \gamma_2(t) := \left(\frac{t^{s+1} - t^s}{t^s - t}\right)^{\frac{1}{s}}, \quad t > 0, t \ne 1. \tag{3.15}$$

These functions are extended by continuity at $t = 0$ and $t = 1$. We sketch the graphs of these two functions for $s = 6$ in Figure 3.2.

The following two lemmas will be crucial for our final argument.

Lemma 3.8. *For every* $s > 1$, *the function* γ_1 *is convex and the function* γ_2 *is concave.*

Proof. For γ_1, one can readily check that

$$\gamma_1''(t) = \frac{2st^{s-2}}{(t^s - 1)^3}\beta_1(t)$$

where

$$\beta_1(t) = (s-1)(t^{s+1} - 1) - (s+1)(t^s - t).$$

Next we observe that
$$\beta_1'(t) = (s+1)\beta_2(t),$$
where
$$\beta_2(t) = (s-1)t^s - st^{s-1} + 1$$
and observe that
$$\beta_2'^{s-1} - t^{s-2} = s(s-1)t^{s-2}(t-1).$$
The sign of β_2' is then easily determined, showing that β_2 has a point of global minimum at $t = 1$. Hence $\beta_2(t) \geq \beta_2(1) = 0$. This implies that β_1 is strictly increasing. Since $\beta_1(1) = 0$ we see that the sign of β_1 is the same as the sign of $(t^s - 1)^3$. This means that $\gamma_1''(t) > 0$ for all $t > 0$. At $t = 1$ the limit is $\frac{s^2-1}{3s} > 0$ also.

In order to deal with γ_2, we rewrite it as
$$\gamma_2(t) = \left(\frac{t^r(t-1)}{t^r - 1}\right)^{\frac{1}{r+1}} = \theta(t)^{\frac{1}{r+1}},$$
where $r := s - 1 > 0$. Since
$$\gamma_2''(t) = \frac{1}{(r+1)\theta(t)^{\frac{2r+1}{r+1}}}\left(\theta(t)\theta''(t) - \frac{r}{r+1}\theta'^2\right),$$
we have to show that
$$\delta(t) := \theta(t)\theta''(t) - \frac{r}{r+1}\theta'^2 < 0$$
for all $t > 0$.

The first and second derivatives of θ are given by
$$\theta'(t) = \frac{t^{2r} - (r+1)t^r + rt^{r-1}}{(t^r-1)^2},$$
and
$$\theta''(t) = \frac{r[(r-1)t^{2r-1} - (r+1)t^{2r-2} + (r+1)t^{r-1} - (r-1)t^{r-2}]}{(t^r-1)^3}.$$

These two expressions substituted into $\delta(t)$ yield
$$\delta(t) = -\frac{rt^{2r-2}}{(r+1)}\delta_1(t),$$
where the sign of δ is determined by
$$\delta_1(t) := t^{2r+2} - (t^{r+2} + t^r)(r+1)^2 + t^{r+1}(2r^2 + 4r) + 1.$$
However, $\delta_1(1) = 0$ and $\delta_1'^{r-1}\delta_2(t)$, where
$$\delta_2(t) = 2t^{r+2} - ((r+2)t^2 + r)(r+1) + (2r^2 + 4r)t.$$
Again, observe that $\delta_2(1) = 0$ and $\delta_2'(t) = 2(r+2)\delta_3(t)$, where $\delta_3(t) = t^{r+1} - (r+1)t + r$. Finally, $\delta_3(1) = 0$ and $\delta_3'^{r} - 1)$. Now δ_3 has only a single critical point at $t = 1$ which is a global minimum. Thus $\delta_3(t) \geq \delta_3(1) = 0$. This shows that δ_2 is strictly increasing on $(0, \infty)$ and is zero at $t = 1$. Therefore, $\delta_1(t)$ has a minimum at $t = 1$ implying that $\delta_1(t) \geq 0$ with its only zero at $t = 1$. Hence $\delta(t) < 0$ for all $t \neq 1$. This, and $\lim_{t\to 1}\delta(t) = -\frac{(r+1)(r+2)}{12r^2}$ show that γ_2 is a concave function and completes the proof. $\square$

We shall need the following well-known result which may be formulated with weaker hypotheses. For convenience, we include it here.

Lemma 3.9. *The graphs of two functions f and g twice differentiable on $[a, b]$, f convex ($f'' > 0$) and g concave ($g'' < 0$) cannot have more than two points of intersection.*

Proof. Suppose by way of contradiction that they have at least three points of intersection. We thus assume $(x_1, f(x_1)) = (x_1, g(x_1))$, $(x_2, f(x_2)) = (x_2, g(x_2))$, $(x_3, f(x_3) = (x_3, g(x_3))$, with $a \le x_1 < x_2 < x_3 \le b$ are such points. Next, we look at the expression

$$E = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{g(x_2) - g(x_1)}{x_2 - x_1} - \frac{g(x_3) - g(x_2)}{x_3 - x_2}.$$

By the Mean Value Theorem applied twice to f and f' the expression E is equal to

$$E = f'(c_1) - f'(c_2) = f''(c)(c_1 - c_2) < 0, c_1 \in (x_1, x_2), c_2 \in (x_2, x_3), c \in (c_1, c_2)$$

and applied to g and g' gives

$$E = g'(\xi_1) - g'(\xi_2) = g''(\xi)(\xi_1 - \xi_2) > 0, \xi_1 \in (x_1, x_2), \xi_2 \in (x_2, x_3), \xi \in (\xi_1, \xi_2)$$

which is a contradiction. □

Let us observe that if x_0 is a solution of the equation $\phi(x_0) = 1$ then $(1/x_0, x_0)$ is a solution of the system (3.9).

Theorem 3.10. *If $q \neq 1/3$, then the only critical points of h are $(1, 1)$, $(x_0, 1)$, $(1, \frac{1}{x_0})$, $(\frac{1}{x_0}, x_0)$, where x_0 is as in Theorem 3.5. If $q = 1/3$, $(1, 1)$ is the only critical point.*

Proof. Start with $q = 1/3$. Then Lemma 3.4 and Theorem 3.5 imply the claim that $(1, 1)$ is the only critical point.

Next, for $q \neq 1/3$, we consider the following system in the variables x and y:

$$\begin{cases} \dfrac{1}{x} = \dfrac{(y-1)(1+y^s)}{y^s - 1} \\ \dfrac{1}{x} = \left(\dfrac{y^{s+1} - y^s}{y^s - y}\right)^{\frac{1}{s}}. \end{cases} \tag{3.16}$$

In what follows next we show that every solution of (C_4) is a solution of (3.16). Indeed, if (x, y) is in (C_4), then it satisfies

$$x = \frac{(x^s + 1)(x - 1)}{y(x^s - 1)}, \qquad x = \frac{y^s - 1}{(y - 1)(1 + y^s)}.$$

This implies that

$$\frac{(x^s + 1)(x - 1)}{y(x^s - 1)} = \frac{y^s - 1}{(y - 1)(1 + y^s)},$$

or

$$(x^s + 1)x(y - 1)(1 + y^s) - (x^s + 1)(y - 1)(1 + y^s) = y(x^s - 1)(y^s - 1).$$

Now, use $x(y - 1)(y^s + 1) = y^s - 1$ to simplify the first term of the previous equality and derive

$$(x^s + 1)(y^s - 1) - (x^s + 1)(y - 1)(1 + y^s) - y(x^s - 1)(y^s - 1) = 0.$$

Finally, we solve for x^s to obtain

$$x^s(y^s - 1 - y^{s+1} - y + 1 + y^s - y^{s+1} + y) = y + y^{s+1} - 1 - y^s + y - y^{s+1} - y^s + 1,$$

which is equivalent to

$$x^s(2y^s - 2y^{s+1}) = 2y - 2y^s.$$

So, if $y \neq 1$ this implies $x^s = \frac{y^s - y}{y^{s+1} - y^s}$ which implies that $\frac{1}{x} = \left(\frac{y^{s+1} - y^s}{y^s - y}\right)^{1/s}$.

We observe that $(1, 1/x_0)$, $(1/x_0, x_0)$ are solutions of (3.16). By Lemmas 3.8 and 3.9, these two points are the only solutions of this system, which proves our theorem. □

Using Lemma 3.8 and Theorem 3.10 we infer the next result.

Theorem 3.11. *The point in* $(1/x_0, x_0)$ *in* (C_4) *is not a minimum point.*

Proof. Since at this point, $A = 2x_0 - 1$, $B = 1/x_0$, $C = 1/x_0$ we see that $ABC = \frac{2x_0-1}{x_0^2}$ and the discriminant D takes the same form as in Theorem 3.7. Hence the proof here follows in the same way as in Theorem 3.7. □

Putting together Lemmas 3.2, 3.4, and Theorems 3.7, 3.10, and 3.11, we infer the truth of Theorem 3.1.

In terms of our original problem, we have obtained the following theorem.

Theorem 3.12. *Given the points* D, E, F *on the sides of a triangle* ABC*, and* F_0, F_1, F_2, F_3 *the areas as in Figure 1.1, then*

$$F_0 \geq C_p M_p(F_1, F_2, F_3),$$

where $C_p = \min\left(1, 2\left(\frac{3}{2}\right)^{1/p}\right)$*, for all* $p < 0$.

Proof. We know from [4] that

$$\frac{F_0}{F_1} = \frac{1}{z} + x - 1 = g(z, x), \qquad \frac{F_0}{F_2} = \frac{1}{x} + y - 1 = g(x, y),$$
$$\frac{F_0}{F_3} = \frac{1}{y} + z - 1 = g(y, z).$$

We showed that $f(x, y, z) = g(x, y)^q + g(y, z)^q + g(z, x)^q)$ has the minimum $m = \min(3, 2^{q+1})$. Hence

$$F_0^q(F_1^{-q} + F_2^{-q} + F_3^{-q}) \geq \min(3, 2^{q+1}).$$

This is equivalent to

$$\max(3^{-1}, 2^{-q-1})(F_1^{-q} + F_2^{-q} + F_3^{-q}) \geq F_0^{-q}.$$

Raising this to power $\frac{1}{p} < 0$ ($p = -q$), we get

$$\min(3^{-1/p}, 2^{1-1/p})(F_1^p + F_2^p + F_3^p)^{\frac{1}{p}} \leq F_0.$$

This gives $C_p = \min\left(1, 2\left(\frac{3}{2}\right)^{1/p}\right)$. □

REFERENCES

[1] O. BOTTEMA, R.Z. DJORDJEVIĆ, R.R. JANIC, D.S. MITRINOVIĆ, P.M. VASIC, *Geometric Inequalities*, Groningen, Wolters–Noordhoff, 1969.

[2] P.S. BULLEN, *Handbook of Means and Their Inequalities*, Kluwer, Dordrecht, 2003.

[3] J.J. DUISTERMAT AND A.C. KOLK, *Multidimensional Real Analysis I (Differentiation)*, Cambridge Studies in Advanced Mathematics Vol. 86, Cambridge University Press, New York, 2004.

[4] W. JANOUS, A short note on the Erdös-Debrunner inequality, *Elem. Math.*, **61** (2006), 32–35.

[5] V. MASCIONI, On the Erdös-Debrunner inequality, *J. Ineq. Pure Appl. Math.*, **8**(2) (2007), Art. 32. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=846`].