Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 4, Article 128, 2005

ON OSTROWSKI-GRÜSS-ČEBYŠEV TYPE INEQUALITIES FOR FUNCTIONS WHOSE MODULUS OF DERIVATIVES ARE CONVEX

B.G. PACHPATTE

57 SHRI NIKETAN COLONY
NEAR ABHINAY TALKIES
AURANGABAD 431 001
(MAHARASHTRA) INDIA
bgpachpatte@hotmail.com

Received 17 August, 2005; accepted 30 August, 2005
Communicated by I. Gavrea

ABSTRACT. The aim of the present paper is to establish some new Ostrowski-Grüss-Čebyšev type inequalities involving functions whose modulus of the derivatives are convex.

Key words and phrases: Ostrowski-Grüss-Čebyšev type inequalities, Modulus of derivatives, Convex, Log-convex, Integral identities.

2000 *Mathematics Subject Classification.* 26D15, 26D20.

1. INTRODUCTION

In 1938, A.M. Ostrowski [5] proved the following classial inequality.

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b) i.e., $|f'(x)| \leq M < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) M, \tag{1.1}$$

for all $x \in [a,b]$, where M is a constant.

For two absolutely continuous functions $f, g : [a,b] \to \mathbb{R}$, consider the functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x) g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx \right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx \right), \tag{1.2}$$

provided the involved integrals exist. In 1882, P.L.Čebyšev [6] proved that, if $f', g' \in L_\infty[a,b]$, then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{1.3}$$

ISSN (electronic): 1443-5756
© 2005 Victoria University. All rights reserved.
245-05

In 1934, G. Grüss [6] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.4}$$

provided m, M, n, N are real numbers satisfying the condition $-\infty < m \leq f(x) \leq M < \infty$, $-\infty < n \leq g(x) \leq N < \infty$, for all $x \in [a,b]$.

During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1] – [10] and the references cited therein. Motivated by the recent results given in [1] – [3], in the present paper, we establish some inequalities similar to those given by Ostrowski, Grüss and Čebyšev, involving functions whose modulus of derivatives are convex. The analysis used in the proofs is elementary and based on the use of integral identities proved in [1] and [2].

2. Statement of Results

Let I be a suitable interval of the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$. A function $f : I \to (0,\infty)$ is said to be log-convex, if

$$f(tx + (1-t) y) \leq [f(x)]^t [f(y)]^{1-t},$$

for all $x, y \in I$ and $t \in [0,1]$ (see [10]). We need the following identities proved in [1] and [2] respectively:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 f'[(1-\lambda)x + \lambda t]\,dt\right]dt,$$

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + (x-a)^2\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a + \lambda x]\,d\lambda - (b-x)^2\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x + (1-\lambda) b]\,d\lambda,$$

for $x \in [a,b]$ where $f : [a,b] \to \mathbb{R}$ is an absolutely continuous function on $[a,b]$ and $\lambda \in [0,1]$.

We use the following notation to simplify the details of presentation:

$$S(f,g) = f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)\,dt + f(x)\int_a^b g(t)\,dt\right],$$

and define $\|\cdot\|_\infty$ as the usual Lebesgue norm on $L_\infty[a,b]$ i.e., $\|h\|_\infty := \operatorname{ess\,sup}_{t\in[a,b]} |h(t)|$ for $h \in L_\infty[a,b]$.

The following theorems deal with Ostrowski type inequalities involving two functions.

Theorem 2.1. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$.*

(a_1) *If $|f'|, |g'|$ are convex on $[a,b]$, then*

$$|S(f,g)| \leq \frac{1}{4}\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right](b-a) \times \{|g(x)|\,[|f'(x)| + \|f'\|_\infty] + |f(x)|\,[|g'(x)| + \|g'\|_\infty]\}, \tag{2.1}$$

for $x \in [a,b]$.

(a_2) *If* $|f'|, |g'|$ *are log-convex on* $[a,b]$*, then*

$$|S(f,g)| \le \frac{1}{2(b-a)}\left\{|g(x)||f'(x)|\int_a^b |x-t|\left(\frac{A-1}{\log A}\right)dt + |f(x)||g'(x)|\int_a^b |x-t|\left(\frac{B-1}{\log B}\right)dt\right\}, \tag{2.2}$$

for $x\in[a,b]$*, where*

$$A = \frac{|f'(t)|}{|f'(x)|}, B = \frac{|g'(t)|}{|g'(x)|}. \tag{2.3}$$

Theorem 2.2. *Let* $f,g:[a,b]\to\mathbb{R}$ *be absolutely continuous functions on* $[a,b]$.

(b_1) *If* $|f'|, |g'|$ *are convex on* $[a,x]$ *and* $[x,b]$*, then*

$$|S(f,g)| \le \frac{1}{2}\{|g(x)|F(x) + |f(x)|G(x)\}, \tag{2.4}$$

for $x\in[a,b]$*, where*

$$F(x) = \frac{1}{6}\Bigg[|f'(a)|\left(\frac{x-a}{b-a}\right)^2 + |f'(b)|\left(\frac{b-x}{b-a}\right)^2 + \left\{1+4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}|f'(x)|\Bigg](b-a), \tag{2.5}$$

$$G(x) = \frac{1}{6}\Bigg[|g'(a)|\left(\frac{x-a}{b-a}\right)^2 + |g'(b)|\left(\frac{b-x}{b-a}\right)^2 + \left\{1+4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}|g'(x)|\Bigg](b-a), \tag{2.6}$$

for $x\in[a,b]$.

(b_2) *If* $|f'|, |g'|$ *are log-convex on* $[a,x]$ *and* $[x,b]$*, then*

$$|S(f,g)| \le |g(x)|H(x) + |f(x)|L(x), \tag{2.7}$$

for $x\in[a,b]$*, where*

$$H(x) = \frac{1}{2}(b-a)\left[|f'(a)|\left(\frac{x-a}{b-a}\right)^2\frac{A_1\log A_1+1-A_1}{(\log A_1)^2} + |f'(b)|\left(\frac{b-x}{b-a}\right)^2\frac{B_1\log B_1+1-B_1}{(\log B_1)^2}\right], \tag{2.8}$$

$$L(x) = \frac{1}{2}(b-a)\left[|g'(a)|\left(\frac{x-a}{b-a}\right)^2\frac{A_2\log A_2+1-A_2}{(\log A_2)^2} + |g'(b)|\left(\frac{b-x}{b-a}\right)^2\frac{B_2\log B_2+1-B_2}{(\log B_2)^2}\right], \tag{2.9}$$

and

$$A_1 = \frac{|f'(x)|}{|f'(a)|}, \quad B_1 = \frac{|f'(x)|}{|f'(b)|}, \tag{2.10}$$

$$A_2 = \frac{|g'(x)|}{|g'(a)|}, \quad B_2 = \frac{|g'(x)|}{|g'(b)|}, \tag{2.11}$$

for $x \in [a,b]$.

The Grüss type inequalities are embodied in the following theorems.

Theorem 2.3. *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions on* $[a,b]$.

(c_1) *If* $|f'|, |g'|$ *are convex on* $[a,b]$*, then*

$$\begin{aligned} &|T(f,g)| \\ &\quad \le \frac{1}{4(b-a)^2} \int_a^b [|g(x)| [|f'(x)| + \|f'\|_\infty] + |f(x)| [|g'(x)| + \|g'\|_\infty]] E(x)\, dx, \end{aligned} \tag{2.12}$$

where

$$E(x) = \frac{(x-a)^2 + (b-x)^2}{2}, \tag{2.13}$$

for $x \in [a,b]$.

(c_2) *If* $|f'|, |g'|$ *are log-convex on* $[a,b]$*, then*

$$\begin{aligned} |T(f,g)| \le \frac{1}{2(b-a)^2} \int_a^b \Bigg[|g(x)| \int_a^b |x-t| \, |f'(x)| \left(\frac{A-1}{\log A} \right) dt \\ + |f(x)| \int_a^b |x-t| \, |g'(x)| \left(\frac{B-1}{\log B} \right) dt \Bigg] dx, \end{aligned} \tag{2.14}$$

where A*,* B *are defined by (2.3).*

Theorem 2.4. *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions on* $[a,b]$.

(d_1) *If* $|f'|, |g'|$ *are convex on* $[a,b]$*, then*

$$\begin{aligned} |T(f,g)| \le \frac{1}{2} \int_a^b \Bigg[\left(\frac{x-a}{b-a} \right)^2 \Bigg[|g(x)| \left\{ \frac{1}{6} |f'(a)| + \frac{1}{3} |f'(x)| \right\} \\ + |f(x)| \left\{ \frac{1}{6} |g'(a)| + \frac{1}{3} |g'(x)| \right\} \Bigg] \\ + \left(\frac{b-x}{b-a} \right)^2 \Bigg[|g(x)| \left\{ \frac{1}{3} |f'(x)| + \frac{1}{6} |f'(b)| \right\} \\ + |f(x)| \left\{ \frac{1}{3} |g'(x)| + \frac{1}{6} |g'(b)| \right\} \Bigg] \Bigg] dx, \end{aligned} \tag{2.15}$$

(d_2) *If* $|f'|, |g'|$ *are log-convex on* $[a,x]$ *and* $[x,b]$*, then*

$$|T(f,g)| \leq \frac{1}{2}\int_a^b \left[\left(\frac{x-a}{b-a}\right)^2 \left\{|g(x)|\,|f'(a)|\,\frac{A_1\log A_1+1-A_1}{(\log A_1)^2} + |f(x)|\,|g'(a)|\,\frac{A_2\log A_2+1-A_2}{(\log A_2)^2}\right\} + \left(\frac{b-x}{b-a}\right)^2 \left\{|g(x)|\,|f'(b)|\,\frac{B_1\log B_1+1-B_1}{(\log B_1)^2} + |f(x)|\,|g'(b)|\,\frac{B_2\log B_2+1-B_2}{(\log B_2)^2}\right\}\right] dx, \tag{2.16}$$

where A_1, B_1 *and* A_2, B_2 *are defined by (2.10) and (2.11).*

The next theorem contains Čebyšev type inequalities.

Theorem 2.5. *Let* $f, g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions on* $[a,b]$.

(e_1) *If* $|f'|, |g'|$ *are convex on* $[a,b]$*, then*

$$|T(f,g)| \leq \frac{1}{4(b-a)^3}\int_a^b \left[|f'(x)| + \|f'\|_\infty\right]\left[|g'(x)| + \|g'\|_\infty\right] E^2(x)\,dx, \tag{2.17}$$

where $E(x)$ *is given by (2.13).*

(e_2) *If* $|f'|, |g'|$ *are log-convex on* $[a,b]$*, then*

$$|T(f,g)| \leq \frac{1}{(b-a)^3}\int_a^b \left[\left\{\int_a^b |x-t|\,|f'(x)|\left(\frac{A-1}{\log A}\right)dt\right\} \times \left\{\int_a^b |x-t|\,|g'(x)|\left(\frac{B-1}{\log B}\right)dt\right\}\right] dx, \tag{2.18}$$

where A, B *are defined by (2.3).*

3. PROOFS OF THEOREMS 2.1 AND 2.2

Proof of Theorem 2.1. From the hypotheses of Theorem 2.1, the following identities hold:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 f'[(1-\lambda)x+\lambda t]d\lambda\right] dt, \tag{3.1}$$

$$g(x) = \frac{1}{b-a}\int_a^b g(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 g'[(1-\lambda)x+\lambda t]d\lambda\right] dt, \tag{3.2}$$

for $x \in [a,b]$. Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting we have

$$S(f,g) = \frac{1}{2(b-a)}\left\{g(x)\int_a^b (x-t)\left[\int_0^1 f'[(1-\lambda)x+\lambda t]d\lambda\right] dt + f(x)\int_a^b (x-t)\left[\int_0^1 g'[(1-\lambda)x+\lambda t]d\lambda\right] dt\right\}. \tag{3.3}$$

(a_1) Since $|f'|, |g'|$ are convex on $[a,b]$, from (3.3) we observe that

$$\begin{aligned}
|S(f,g)| &\le \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[\int_0^1 |f'[(1-\lambda)x+\lambda t]|\,d\lambda\right]dt\right. \\
&\qquad \left.+|f(x)|\int_a^b |x-t|\left[\int_0^1 |g'[(1-\lambda)x+\lambda t]|\,d\lambda\right]dt\right\} \\
&\le \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[\int_0^1 \{(1-\lambda)|f'(x)|+\lambda|f'(t)|\}\,d\lambda\right]dt\right. \\
&\qquad \left.+|f(x)|\int_a^b |x-t|\left[\int_0^1 \{(1-\lambda)|g'(x)|+\lambda|g'(t)|\}\,d\lambda\right]dt\right\} \\
&= \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[|f'(x)|\int_0^1 (1-\lambda)\,d\lambda+|f'(t)|\int_0^1 \lambda d\lambda\right]dt\right. \\
&\qquad \left.+|f(x)|\int_a^b |x-t|\left[|g'(x)|\int_0^1 (1-\lambda)\,d\lambda+|g'(t)|\int_0^1 \lambda d\lambda\right]dt\right\} \\
&= \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\frac{1}{2}\left[|f'(x)|+|f'(t)|\right]dt\right. \\
&\qquad \left.+|f(x)|\int_a^b |x-t|\frac{1}{2}\left[|g'(x)|+|g'(t)|\right]dt\right\} \\
&\le \frac{1}{4(b-a)}\left\{|g(x)| \underset{t\in[a,b]}{ess.\sup}\left[|f'(x)|+|f'(t)|\right]\int_a^b |x-t|\,dt\right. \\
&\qquad \left.+|f(x)| \underset{t\in[a,b]}{ess.\sup}\left[|g'(x)|+|g'(t)|\right]\int_a^b |x-t|\,dt\right\} \\
&= \frac{1}{4(b-a)}\left\{|g(x)|\left[|f'(x)|+\|f'\|_\infty\right]\right. \\
&\qquad \left.+|f(x)|\left[|g'(x)|+\|g'\|_\infty\right]\right\}\int_a^b |x-t|\,dt \\
&= \frac{1}{4}\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right] \\
&\qquad\qquad \times\left\{|g(x)|\left[|f'(x)|+\|f'\|_\infty\right]+|f(x)|\left[|g'(x)|+\|g'\|_\infty\right]\right\} \\
&= \frac{1}{4}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \\
&\qquad\qquad \times(b-a)\left\{|g(x)|\left[|f'(x)|+\|f'\|_\infty\right]+|f(x)|\left[|g'(x)|+\|g'\|_\infty\right]\right\}.
\end{aligned}$$

This is the required inequality in (2.1).

(a_2) Since $|f'|, |g'|$ are log-convex on $[a,b]$, from (3.3) we observe that

$$\begin{aligned}
|S(f,g)| &\le \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[\int_0^1 |f'[(1-\lambda)x+\lambda t]|\,d\lambda\right]dt\right. \\
&\qquad \left.+|f(x)|\int_a^b |x-t|\left[\int_0^1 |g'[(1-\lambda)x+\lambda t]|\,d\lambda\right]dt\right\}
\end{aligned}$$

$$\leq \frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[\int_0^1 [|f'(x)|]^{1-\lambda}[|f'(t)|]^{\lambda}d\lambda\right]dt\right.$$
$$\left.+|f(x)|\int_a^b |x-t|\left[\int_0^1 [|g'(x)|]^{1-\lambda}[|g'(t)|]^{\lambda}d\lambda\right]dt\right\}$$
$$=\frac{1}{2(b-a)}\left\{|g(x)|\int_a^b |x-t|\left[|f'(x)|\int_0^1\left[\frac{|f'(t)|}{|f'(x)|}\right]^{\lambda}d\lambda\right]dt\right.$$
$$\left.+|f(x)|\int_a^b |x-t|\left[|g'(x)|\int_0^1\left[\frac{|g'(t)|}{|g'(x)|}\right]^{\lambda}d\lambda\right]dt\right\}$$
$$=\frac{1}{2(b-a)}\left\{|g(x)|\,|f'(x)|\int_a^b |x-t|\left(\frac{A-1}{logA}\right)dt\right.$$
$$\left.+|f(x)|\,|g'(x)|\int_a^b |x-t|\left(\frac{B-1}{\log B}\right)dt\right\}.$$

This completes the proof of the inequality (2.2). □

Proof of Theorem 2.2. From the hypotheses of Theorem 2.2, the following identities hold:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+(x-a)^2\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a+\lambda x]\,d\lambda$$
$$-(b-x)^2\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x+(1-\lambda)b]\,d\lambda, \tag{3.4}$$

$$g(x)=\frac{1}{b-a}\int_a^b g(t)\,dt+(x-a)^2\frac{1}{b-a}\int_0^1 \lambda g'[(1-\lambda)a+\lambda x]\,d\lambda$$
$$-(b-x)^2\frac{1}{b-a}\int_0^1 \lambda g'[\lambda x+(1-\lambda)b]\,d\lambda. \tag{3.5}$$

Multiplying both sides of (3.4) and (3.5) by $g(x)$ and $f(x)$ respectively, adding the resulting identities and rewriting we have

$$S(f,g)=\frac{1}{2}\left\{g(x)\left[(x-a)^2\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a+\lambda x]\,d\lambda\right.\right.$$
$$\left.-(b-x)^2\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x+(1-\lambda)b]\,d\lambda\right]$$
$$+f(x)\left[(x-a)^2\frac{1}{b-a}\int_0^1 \lambda g'[(1-\lambda)a+\lambda x]\,d\lambda\right.$$
$$\left.\left.-(b-x)^2\frac{1}{b-a}\int_0^1 \lambda g'[\lambda x+(1-\lambda)b]\,d\lambda\right]\right\}. \tag{3.6}$$

(b_1) Since $|f'|,|g'|$ are convex on $[a,x]$ and $[x,b]$, from (3.6) we observe that

$$|S(f,g)|\leq\frac{1}{2}\{|g(x)|\,M(x)+|f(x)|\,N(x)\}, \tag{3.7}$$

where

$$M(x) = (x-a)^2 \frac{1}{b-a} \int_0^1 \lambda \left| f'\left[(1-\lambda)a + \lambda x\right]\right| d\lambda + (b-x)^2 \frac{1}{b-a} \int_0^1 \lambda \left| f'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda, \tag{3.8}$$

$$N(x) = \frac{(x-a)^2}{b-a} \int_0^1 \lambda \left| g'\left[(1-\lambda)a + \lambda x\right]\right| d\lambda + \frac{(b-x)^2}{b-a} \int_0^1 \lambda \left| g'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda. \tag{3.9}$$

Next, we observe that

$$\begin{aligned} \int_0^1 \lambda \left| f'\left[(1-\lambda)a + \lambda x\right]\right| d\lambda &\le \left|f'(a)\right| \int_0^1 \lambda(1-\lambda)\, d\lambda + \left|f'(x)\right| \int_0^1 \lambda^2 d\lambda \\ &= \frac{1}{6}\left|f'(a)\right| + \frac{1}{3}\left|f'(x)\right| \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \int_0^1 \lambda \left| f'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda &\le \left|f'(x)\right| \int_0^1 \lambda^2 d\lambda + \left|f'(b)\right| \int_0^1 \lambda(1-\lambda)\, d\lambda \\ &= \frac{1}{3}\left|f'(x)\right| + \frac{1}{6}\left|f'(b)\right|. \end{aligned} \tag{3.11}$$

Similarly we have

$$\int_0^1 \lambda \left| g'\left[(1-\lambda)a + \lambda x\right]\right| d\lambda \le \frac{1}{6}\left|g'(a)\right| + \frac{1}{3}\left|g'(x)\right|, \tag{3.12}$$

$$\int_0^1 \lambda \left| g'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda \le \frac{1}{3}\left|g'(x)\right| + \frac{1}{6}\left|g'(b)\right|. \tag{3.13}$$

From (3.8), (3.10) and (3.11) we observe that

$$\begin{aligned} M(x) &= \left[\left(\frac{x-a}{b-a}\right)^2 \int_0^1 \lambda \left| f'\left[(1-\lambda)a + \lambda x\right]\right| d\lambda \right. \\ &\qquad \left. + \left(\frac{b-x}{b-a}\right)^2 \int_0^1 \lambda \left| f'\left[\lambda x + (1-\lambda) b\right]\right| d\lambda \right](b-a) \\ &\le \left[\left(\frac{x-a}{b-a}\right)^2 \left\{\frac{1}{6}\left|f'(a)\right| + \frac{1}{3}\left|f'(x)\right|\right\} \right. \\ &\qquad \left. + \left(\frac{b-x}{b-a}\right)^2 \left\{\frac{1}{3}\left|f'(x)\right| + \frac{1}{6}\left|f'(b)\right|\right\}\right](b-a) \end{aligned} \tag{3.14}$$

$$= \frac{1}{6}\left[\left(\frac{x-a}{b-a}\right)^2 |f'(a)| + \left(\frac{b-x}{b-a}\right)^2 |f'(b)|\right](b-a)$$
$$+ \frac{1}{3}\left[\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right] |f'(x)| (b-a)$$
$$= \frac{1}{6}\left[\left(\frac{x-a}{b-a}\right)^2 |f'(a)| + \left(\frac{b-x}{b-a}\right)^2 |f'(b)|\right.$$
$$\left. + 2\left[\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right] |f'(x)|\right](b-a).$$

It is easy to observe that

$$2\left[\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right] = \frac{4}{b-a}\left[\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right] \tag{3.15}$$
$$= \frac{4}{b-a}\left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)$$
$$= \left[1 + 4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right].$$

Using (3.15) in (3.14) we get

$$M(x) \le F(x). \tag{3.16}$$

Similarly, from (3.9), (3.12), (3.13) we get

$$N(x) \le G(x). \tag{3.17}$$

Using (3.16), (3.17) in (3.7) we get the required inequality in (2.4).

(b_2) Since $|f'|, |g'|$ are log-convex on $[a,x]$ and $[x,b]$, from (3.6) we observe that

$$|S(f,g)| \le \frac{1}{2}(b-a)\left\{|g(x)|\left[\left(\frac{x-a}{b-a}\right)^2 \int_0^1 \lambda |f'[(1-\lambda)a + \lambda x]| d\lambda\right.\right. \tag{3.18}$$
$$\left. + \left(\frac{b-x}{b-a}\right)^2 \int_0^1 \lambda |f'[\lambda x + (1-\lambda)b]| d\lambda\right]$$
$$+ |f(x)|\left[\left(\frac{x-a}{b-a}\right)^2 \int_0^1 \lambda |g'[(1-\lambda)a + \lambda x]| d\lambda\right.$$
$$\left.\left. + \left(\frac{b-x}{b-a}\right)^2 \int_0^1 \lambda |g'[\lambda x + (1-\lambda)b]| d\lambda\right]\right\}$$

$$\leq \frac{1}{2}(b-a)\left\{|g(x)|\left[\left(\frac{x-a}{b-a}\right)^2\int_0^1 \lambda\left[|f'(a)|\right]^{1-\lambda}\left[|f'(x)|\right]^{\lambda}d\lambda\right.\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2\int_0^1 \lambda\left[|f'(x)|\right]^{\lambda}\left[|f'(b)|\right]^{1-\lambda}\right]$$
$$+|f(x)|\left[\left(\frac{x-a}{b-a}\right)^2\int_0^1 \lambda\left[|g'(a)|\right]^{1-\lambda}\left[|g'(x)|\right]^{\lambda}d\lambda\right.$$
$$\left.\left.+\left(\frac{b-x}{b-a}\right)^2\int_0^1 \lambda\left[|g'(x)|\right]^{\lambda}\left[|g'(b)|\right]^{1-\lambda}d\lambda\right]\right\}$$
$$=\frac{1}{2}(b-a)\left\{|g(x)|\left[\left(\frac{x-a}{b-a}\right)^2|f'(a)|\int_0^1 \lambda A_1^{\lambda}d\lambda\right.\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2|f'(b)|\int_0^1 \lambda B_1^{\lambda}d\lambda\right]$$
$$+|f(x)|\left[\left(\frac{x-a}{b-a}\right)^2|g'(a)|\int_0^1 \lambda A_2^{\lambda}d\lambda\right.$$
$$\left.\left.+\left(\frac{b-x}{b-a}\right)^2|g'(b)|\int_0^1 \lambda B_2^{\lambda}d\lambda\right]\right\}.$$

A simple calculation shows that for any $C>0$ we have (see [2])

$$\int_0^1 \lambda C^{\lambda}d\lambda=\frac{C\log C+1-C}{(\log C)^2}. \tag{3.19}$$

Using this fact in (3.18) we get the required inequality in (2.7). □

4. PROOFS OF THEOREMS 2.3 AND 2.4

Proof of Theorem 2.3. From the hypotheses of Theorem 2.3 the identities (3.1) – (3.3) hold. Integrating both sides of (3.3) with respect to x from a to b and rewriting we have

$$T(f,g)=\frac{1}{2(b-a)^2}\int_a^b\left[g(x)\int_a^b(x-t)\left[\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right]dt\right.$$
$$\left.+f(x)\int_a^b(x-t)\left[\int_0^1 g'[(1-\lambda)x+\lambda t]\,d\lambda\right]dt\right]dx. \tag{4.1}$$

(c_1) Since $|f'|,|g'|$ are convex on $[a,b]$, from (4.1) we observe that

$$|T(f,g)|\leq\frac{1}{2(b-a)^2}\int_a^b\left[|g(x)|\int_a^b|x-t|\left[\int_0^1[(1-\lambda)|f'(x)|+\lambda|f'(t)|]\,d\lambda\right]dt\right.$$
$$\left.+|f(x)|\int_a^b|x-t|\left[\int_0^1[(1-\lambda)|g'(x)|+\lambda|g'(t)|]\,d\lambda\right]dt\right]dx$$

$$
\begin{aligned}
&= \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)| \int_a^b |x-t| \left[\frac{|f'(x)|+|f'(t)|}{2}\right] dt \right.\\
&\qquad \left. + |f(x)| \int_a^b |x-t| \left[\frac{|g'(x)|+|g'(t)|}{2}\right] dt \right] dx\\
&\le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)| \int_a^b |x-t| \underset{t\in[a,b]}{ess\ \sup} \left[\frac{|f'(x)|+|f'(t)|}{2}\right] dt \right.\\
&\qquad \left. + |f(x)| \int_a^b |x-t| \underset{t\in[a,b]}{ess\ \sup} \left[\frac{|g'(x)|+|g'(t)|}{2}\right] dt \right] dx\\
&= \frac{1}{4(b-a)^2}\int_a^b [|g(x)| [|f'(x)| + \|f'\|_\infty] dt\\
&\qquad + |f(x)| [|g'(x)| + \|g'\|_\infty]] \left\{ \int_a^b |x-t| dt \right\} dx\\
&= \frac{1}{4(b-a)^2}\int_a^b [|g(x)| [|f'(x)| + \|f'\|_\infty] dt\\
&\qquad + |f(x)| [|g'(x)| + \|g'\|_\infty]] E(x)\, dx.
\end{aligned}
$$

This completes the proof of the inequality (2.14).

(c_2) Since $|f'|, |g'|$ are log-convex on $[a,b]$ from (4.1) we observe that

$$
\begin{aligned}
|T(f,g)| &\le \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)| \int_a^b |x-t| \left[\int_0^1 [|f'(x)|]^{1-\lambda} [|f'(t)|]^{\lambda} d\lambda\right] dt \right.\\
&\qquad \left. + |f(x)| \int_a^b |x-t| \left[\int_0^1 [|g'(x)|]^{1-\lambda} [|g'(t)|]^{\lambda} d\lambda\right] dt \right] dx\\
&= \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)| \int_a^b |x-t| \left[|f'(x)| \int_0^1 \left[\frac{|f'(t)|}{|f'(x)|}\right]^{\lambda} d\lambda \right] dt \right.\\
&\qquad \left. + |f(x)| \int_a^b |x-t| \left[|g'(x)| \int_0^1 \left[\frac{|g'(t)|}{|g'(x)|}\right]^{\lambda} d\lambda \right] dt \right] dx\\
&= \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)| \int_a^b |x-t|\, |f'(x)| \left(\frac{A-1}{\log A}\right) dt \right.\\
&\qquad \left. + |f(x)| \int_a^b |x-t|\, |g'(x)| \left(\frac{B-1}{\log B}\right) dt \right] dx,
\end{aligned}
$$

where A, B are defined by (2.3). This is the required inequality in (2.14). □

Proof of Theorem 2.4. From the hypotheses of Theorem 2.4 the identities (3.4) – (3.6) hold. Integrating both sides of (3.6) with respect to x from a to b and rewriting we have

$$
\begin{aligned}
(4.2)\quad T(f,g) = \frac{1}{2}\int_a^b &\left[\left(\frac{x-a}{b-a}\right)^2 \left[g(x) \int_0^1 \lambda f'[(1-\lambda)a + \lambda x]\, d\lambda \right.\right.\\
&\qquad \left.\left. + f(x) \int_0^1 \lambda g'[(1-\lambda)a+\lambda x] d\lambda \right]\right.
\end{aligned}
$$

$$- \left(\frac{b-x}{b-a}\right)^2 \left[g(x) \int_0^1 \lambda f'[\lambda x + (1-\lambda) b]\, d\lambda \right.$$
$$\left. + f(x) \int_0^1 \lambda g'[\lambda x + (1-\lambda) b] d\lambda\right]\Bigg] dx.$$

(d_1) Since $|f'|, |g'|$ are convex on $[a, x]$ and $[x, b]$ from (4.2) we observe that

$$\begin{aligned}
|T(f,g)| \leq \frac{1}{2} \int_a^b \Bigg[& \left(\frac{x-a}{b-a}\right)^2 \left[|g(x)| \int_0^1 \lambda |f'[(1-\lambda) a + \lambda x]|\, d\lambda \right. \\
& \left. + |f(x)| \int_0^1 \lambda |g'[(1-\lambda) a + \lambda x]| d\lambda \right] \\
& + \left(\frac{b-x}{b-a}\right)^2 \left[|g(x)| \int_0^1 \lambda |f'[\lambda x + (1-\lambda) b]|\, d\lambda \right. \\
& \left. + |f(x)| \int_0^1 \lambda |g'[\lambda x + (1-\lambda) b]| d\lambda \right]\Bigg] dx \\
\leq \frac{1}{2} \int_a^b \Bigg[& \left(\frac{x-a}{b-a}\right)^2 \left[|g(x)| \int_0^1 \lambda \{(1-\lambda)|f'(a)| + \lambda |f'(x)|\}\, d\lambda \right. \\
& \left. + |f(x)| \int_0^1 \lambda \{(1-\lambda)|g'(a)| + \lambda |g'(x)|\} d\lambda \right] \\
& + \left(\frac{b-x}{b-a}\right)^2 \left[|g(x)| \int_0^1 \lambda \{\lambda |f'(x)| + (1-\lambda)|f'(b)|\}\, d\lambda \right. \\
& \left. + |f(x)| \int_0^1 \lambda \{\lambda |g'(x)| + (1-\lambda)|g'(b)|\} d\lambda \right]\Bigg] dx \\
= \frac{1}{2} \int_a^b \Bigg[& \left(\frac{x-a}{b-a}\right)^2 \left[|g(x)| \left\{\frac{1}{6}|f'(a)| + \frac{1}{3}|f'(x)|\right\} \right. \\
& \left. + |f(x)| \left\{\frac{1}{6}|g'(a)| + \frac{1}{3}|g'(x)|\right\}\right] \\
& + \left(\frac{b-x}{b-a}\right)^2 \left[|g(x)| \left\{\frac{1}{3}|f'(x)| + \frac{1}{6}|f'(b)|\right\} \right. \\
& \left. + |f(x)| \left\{\frac{1}{3}|g'(x)| + \frac{1}{6}|g'(b)|\right\}\right]\Bigg] dx.
\end{aligned}$$

This proves the inequality in (2.15).

(d_2) Since $|f'|, |g'|$ are log-convex on $[a, x]$ and $[x, b]$, from (4.2) and the fact (3.19) we observe that

$$\begin{aligned}
|T(f,g)| \leq \frac{1}{2} \int_a^b \Bigg[& \left(\frac{x-a}{b-a}\right)^2 \left[|g(x)| \int_0^1 \lambda [|f'(a)|]^{1-\lambda} [|f'(x)|]^{\lambda}\, d\lambda \right. \\
& \left. + |f(x)| \int_0^1 \lambda [|g'(a)|]^{1-\lambda} [|g'(x)|]^{\lambda}\, d\lambda \right]
\end{aligned}$$

$$+\left(\frac{b-x}{b-a}\right)^2\left[|g(x)|\int_0^1\lambda\left[|f'(x)|\right]^{\lambda}\left[|f'(b)|\right]^{1-\lambda}d\lambda\right.$$
$$\left.\left.+|f(x)|\int_0^1\lambda\left[|g'(x)|\right]^{\lambda}\left[|g'(b)|\right]^{1-\lambda}d\lambda\right]\right]dx$$
$$=\frac{1}{2}\int_a^b\left[\left(\frac{x-a}{b-a}\right)^2\left[|g(x)|\,|f'(a)|\int_0^1\lambda A_1^{\lambda}d\lambda+|f(x)|\,|g'(a)|\int_0^1\lambda B_1^{\lambda}d\lambda\right]\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^2\left[|g(x)|\,|f'(b)|\int_0^1\lambda A_2^{\lambda}d\lambda+|f(x)|\,|g'(b)|\int_0^1\lambda B_2^{\lambda}d\lambda\right]\right]dx$$
$$=\frac{1}{2}\int_a^b\left[\left(\frac{x-a}{b-a}\right)^2\left[|g(x)|\,|f'(a)|\,\frac{A_1\log A_1+1-A_1}{(\log A_1)^2}\right.\right.$$
$$\left.+|f(x)|\,|g'(a)|\,\frac{B_1\log B_1+1-B_1}{(\log B_1)^2}\right]$$
$$+\left(\frac{b-x}{b-a}\right)^2\left[|g(x)|\,|f'(b)|\frac{A_2\log A_2+1-A_2}{(\log A_2)^2}\right.$$
$$\left.\left.+|f(x)|\,|g'(b)|\,\frac{B_2\log B_2+1-B_2}{(\log B_2)^2}\right]\right]dx.$$

This is the desired inequality in (2.16). □

5. PROOF OF THEOREM 2.5

From the hypotheses, the identities (3.1) and (3.2) hold. From (3.1) and (3.2) we observe that

$$\left\{f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right\}\left\{g(x)-\frac{1}{b-a}\int_a^b g(t)\,dt\right\}$$
$$=\left\{\frac{1}{b-a}\int_a^b(x-t)\left[\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right\}$$
$$\times\left\{\frac{1}{b-a}\int_a^b(x-t)\left[\int_0^1 g'\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right\}$$

that is,

$$f(x)g(x)-f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)-g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)$$
$$+\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)$$
$$=\left\{\frac{1}{b-a}\int_a^b(x-t)\left[\int_0^1 f'\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right\}$$
$$\times\left\{\frac{1}{b-a}\int_a^b(x-t)\left[\int_0^1 g'\left[(1-\lambda)x+\lambda t\right]d\lambda\right]dt\right\}. \tag{5.1}$$

Integrating both sides of (5.1) with respect to x from a to b and rewriting we have

$$T(f,g) = \frac{1}{b-a}\int_a^b \left\{ \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 f'[(1-\lambda)x+\lambda t]\,d\lambda\right] dt \right\} \times \left\{ \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 g'[(1-\lambda)x+\lambda t]\,d\lambda\right] dt \right\} dx. \tag{5.2}$$

(e_1) Since $|f'|, |g'|$ are convex on $[a,b]$, from (5.2) we observe that

$$\begin{aligned} |T(f,g)| &\le \frac{1}{b-a}\int_a^b \left\{ \frac{1}{b-a}\int_a^b |x-t|\left[\int_0^1 |f'[(1-\lambda)x+\lambda t]|\,d\lambda\right] dt \right\} \\ &\quad \times \left\{ \frac{1}{b-a}\int_a^b |x-t|\left[\int_0^1 |g'[(1-\lambda)x+\lambda t]|\,d\lambda\right] dt \right\} dx \\ &\le \frac{1}{(b-a)^3}\int_a^b \left\{ \int_a^b |x-t|\left[\int_0^1 [(1-\lambda)|f'(x)|+\lambda|f'(t)|]\,d\lambda\right] dt \right\} \\ &\quad \times \left\{ \int_a^b |x-t|\left[\int_0^1 [(1-\lambda)|g'(x)|+\lambda|g'(t)|]\,d\lambda\right] dt \right\} dx \\ &= \frac{1}{(b-a)^3}\int_a^b \left\{ \int_a^b |x-t|\left[\frac{|f'(x)|+|f'(t)|}{2}\right] dt \right\} \\ &\quad \times \left\{ \int_a^b |x-t|\left[\frac{|g'(x)|+|g'(t)|}{2}\right] dt \right\} dx. \end{aligned}$$

The rest of the proof of inequality (2.17) can be completed by closely looking at the proof of Theorem 2.3, part (c_1).

(e_2) The proof follows by closely looking at the proof of (e_1) given above and the proof of Theorem 2.3, part (c_2) . We omit the details.

REFERENCES

[1] N.S. BARNETT, P. CERONE, S.S. DRAGOMIR, M.R. PINHEIRO AND A. SOFO, Ostrowski type inequalities for functions whose modulus of derivatives are convex and applications, *RGMIA Res. Rep. Coll.*, **5**(2) (2002), 219–231. [ONLINE: `http://rgmia.vu.edu.au/v5n2.html`]

[2] P.CERONE AND S.S.DRAGOMIR, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.*, **37**(2) (2004), 299–308.

[3] S.S.DRAGOMIR AND A.SOFO, Ostrowski type inequalities for functions whose derivatives are convex, *Proceedings of the 4th International Conference on Modelling and Simulation,* November 11-13, 2002. Victoria University, Melbourne, Australia. *RGMIA Res. Rep. Coll.*, **5**(Supp) (2002), Art. 30. [ONLINE: `http://rgmia.vu.edu.au/v5(E).html`]

[4] S.S.DRAGOMIR AND Th.M. RASSIAS (Eds.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[5] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[6] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[7] B.G. PACHPATTE, A note on integral inequalities involving two log-convex functions, *Math. Inequal. Appl.*, **7**(4) (2004), 511–515.

[8] B.G. PACHPATTE, A note on Hadamard type integral inequalities involving several log-convex functions, *Tamkang J. Math.*, **36**(1) (2005), 43–47.

[9] B.G. PACHPATTE, *Mathematical Inequalities*, North-Holland Mathematical Library, Vol. 67 Elsevier, 2005.

[10] J.E. PEČARIĆ, F. PROSCHAN AND Y.L. TANG, *Convex functions, Partial Orderings and Statistcal Applications*, Academic Press, New York, 1991.