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COEFFICIENT INEQUALITIES FOR CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT. In view of classes of uniformly starlike and convex functions in the open unit disc $\mathbb{U}$ which was considered by S. Shams, S.R. Kulkarni and J.M. Jahangiri, some coefficient inequalities for functions are discussed.

Key words and phrases: Uniformly starlike, Uniformly convex.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Let $\mathcal{S}^*(\beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathbb{U}) \tag{1.2}$$

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for some β $(0 \leqq \beta < 1)$. A function $f(z) \in \mathcal{S}^*(\beta)$ is said to be starlike of order β in $\mathbb{U}$. Also let $\mathcal{K}(\beta)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some β $(0 \leqq \beta < 1)$. A function $f(z)$ in $\mathcal{K}(\beta)$ is said to be convex of order β in $\mathbb{U}$. In view of the class $\mathcal{S}^*(\beta)$, Shams, Kulkarni and Jahangiri [3] have introduced the subclass $\mathcal{SD}(\alpha, \beta)$ of $\mathcal{A}$ consisting of functions $f(z)$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some $\alpha \geqq 0$ and β $(0 \leqq \beta < 1)$. We also denote by $\mathcal{KD}(\alpha, \beta)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \left|\frac{zf''(z)}{f'(z)}\right| + \beta \qquad (z \in \mathbb{U}) \tag{1.5}$$

for some $\alpha \geqq 0$ and β $(0 \leqq \beta < 1)$. Then we note that $f(z) \in \mathcal{KD}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha, \beta)$. For such classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, Shams, Kulkarni and Jahangiri [3] have shown some sufficient conditions for $f(z)$ to be in the classes $\mathcal{SD}(\alpha, \beta)$ or $\mathcal{KD}(\alpha, \beta)$.

2. COEFFICIENT INEQUALITIES

Our first result is contained in

Theorem 2.1. *If $f(z) \in \mathcal{SD}(\alpha, \beta)$ with $0 \leqq \alpha \leqq \beta$ or $\alpha > \frac{1+\beta}{2}$ then $f(z) \in \mathcal{S}^*\left(\frac{\beta-\alpha}{1-\alpha}\right)$.*

Proof. Since $\operatorname{Re}(w) \leqq |w|$ for any complex number w, $f(z) \in \mathcal{SD}(\alpha, \beta)$ implies that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha\ \operatorname{Re}\left(\frac{zf'(z)}{f(z)} - 1\right) + \beta, \tag{2.1}$$

or that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{\beta - \alpha}{1 - \alpha} \qquad (z \in \mathbb{U}). \tag{2.2}$$

If $0 \leqq \alpha \leqq \beta$, then we have that

$$0 \leqq \frac{\beta - \alpha}{1 - \alpha} < 1,$$

and if $\alpha > \dfrac{1+\beta}{2}$, then we have

$$-1 < \frac{\alpha - \beta}{\alpha - 1} \leqq 0.$$

□

Corollary 2.2. *If $f(z) \in \mathcal{KD}(\alpha, \beta)$ with $0 \leqq \alpha \leqq \beta$ or $\alpha > \frac{1+\beta}{2}$, then $f(z) \in \mathcal{K}\left(\frac{\beta-\alpha}{1-\alpha}\right)$.*

Next we derive

Theorem 2.3. *If $f(z) \in \mathcal{SD}(\alpha, \beta)$, then*

$$|a_2| \leqq \frac{2(1-\beta)}{|1-\alpha|} \tag{2.3}$$

and

$$|a_n| \leqq \frac{2(1-\beta)}{(n-1)\,|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j\,|1-\alpha|}\right) \qquad (n \geqq 3). \tag{2.4}$$

Proof. Note that, for $f(z) \in \mathcal{SD}(\alpha,\beta)$,

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{\beta-\alpha}{1-\alpha} \qquad (z \in \mathbb{U}).$$

If we define the function $p(z)$ by

$$p(z) = \frac{(1-\alpha)\frac{zf'(z)}{f(z)} - (\beta-\alpha)}{1-\beta} \qquad (z \in \mathbb{U}), \tag{2.5}$$

then $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ $(z \in \mathbb{U})$. Letting $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, we have

$$zf'(z) = f(z)\left(1+\frac{1-\beta}{1-\alpha}\sum_{n=1}^{\infty} p_n z^n\right). \tag{2.6}$$

Therefore, (2.6) implies that

$$(n-1)a_n = \frac{1-\beta}{1-\alpha}(p_{n-1} + a_2 p_{n-2} + \cdots + a_{n-1}p_1). \tag{2.7}$$

Applying the coefficient estimates such that $|p_n| \leqq 2$ $(n \geqq 1)$ (see [1]) for Carathéodory functions, we obtain that

$$|a_n| \leqq \frac{2(1-\beta)}{(n-1)\,|1-\alpha|}(1 + |a_2| + |a_3| + \cdots + |a_{n-1}|). \tag{2.8}$$

Therefore, for $n = 2$,

$$|a_2| \leqq \frac{2(1-\beta)}{|1-\alpha|},$$

which proves (2.3), and, for $n = 3$,

$$|a_3| \leqq \frac{2(1-\beta)}{2\,|1-\alpha|}\left(1+\frac{2(1-\beta)}{|1-\alpha|}\right).$$

Thus, (2.4) holds true for $n = 3$.

Supposing that (2.4) is true for $n = 3, 4, 5, \ldots, k$, we see that

$$\begin{aligned}
|a_{k+1}| \leqq \frac{2(1-\beta)}{k\,|1-\alpha|}\Bigg\{1 &+ \frac{2(1-\beta)}{|1-\alpha|} + \frac{2(1-\beta)}{2\,|1-\alpha|}\left(1+\frac{2(1-\beta)}{|1-\alpha|}\right) \\
&+ \cdots + \frac{2(1-\beta)}{(k-1)\,|1-\alpha|}\prod_{j=1}^{k-2}\left(1+\frac{2(1-\beta)}{j\,|1-\alpha|}\right)\Bigg\} \\
= \frac{2(1-\beta)}{k|1-\alpha|}\prod_{j=1}^{k-1}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)&.
\end{aligned}$$

Consequently, using mathematical induction, we have proved that (2.4) holds true for any $n \geqq 3$. □

Remark 2.4. If we take $\alpha = 0$ in Theorem 2.3, then we have

$$|a_n| \leqq \frac{\prod_{j=2}^{n}(j-2\beta)}{(n-1)!} \qquad (n \geqq 2)$$

which was given by Robertson [2].

Since $f(z) \in \mathcal{KD}(\alpha,\beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha,\beta)$, we have

Corollary 2.5. *If* $f(z) \in \mathcal{KD}(\alpha,\beta)$*, then*

$$|a_2| \leqq \frac{1-\beta}{|1-\alpha|} \tag{2.9}$$

and

$$|a_n| \leqq \frac{2(1-\beta)}{n(n-1)\,|1-\alpha|}\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j\,|1-\alpha|}\right) \qquad (n \geqq 3). \tag{2.10}$$

Remark 2.6. Letting $\alpha = 0$ in Corollary 2.5, we see that

$$|a_n| \leqq \frac{\prod_{j=2}^{n}(j-2\beta)}{n!} \qquad (n \geqq 2),$$

given by Robertson [2].

Further applying Theorem 2.3 we derive:

Theorem 2.7. *If* $f(z) \in \mathcal{SD}(\alpha,\beta)$*, then*

$$\max\left\{0, |z| - \frac{2(1-\beta)}{|1-\alpha|}|z|^2 - \sum_{n=3}^{\infty}\frac{2(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^n\right\}$$
$$\leqq |f(z)| \leqq |z| + \frac{2(1-\beta)}{|1-\alpha|}|z|^2 + \sum_{n=3}^{\infty}\frac{2(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^n$$

and

$$\max\left\{0, 1 - \frac{4(1-\beta)}{|1-\alpha|}|z| - \sum_{n=3}^{\infty}\frac{2n(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^{n-1}\right\}$$
$$\leqq |f'(z)| \leqq 1 + \frac{4(1-\beta)}{|1-\alpha|}|z| + \sum_{n=3}^{\infty}\frac{2n(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^{n-1}.$$

Corollary 2.8. *If* $f(z) \in \mathcal{KD}(\alpha,\beta)$*, then*

$$\max\left\{0, |z| - \frac{1-\beta}{|1-\alpha|}|z|^2 - \sum_{n=3}^{\infty}\frac{2(1-\beta)}{n(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^n\right\}$$
$$\leqq |f(z)| \leqq |z| + \frac{1-\beta}{|1-\alpha|}|z|^2 + \sum_{n=3}^{\infty}\frac{2(1-\beta)}{n(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-2}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^n$$

and

$$\max\left\{0, 1-\frac{2(1-\beta)}{|1-\alpha|}|z| - \sum_{n=3}^{\infty}\frac{2(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^{n-1}\right\}$$
$$\leqq |f'(z)| \leqq 1+\frac{2(1-\beta)}{|1-\alpha|}|z| + \sum_{n=3}^{\infty}\frac{2(1-\beta)}{(n-1)|1-\alpha|}\left(\prod_{j=1}^{n-1}\left(1+\frac{2(1-\beta)}{j|1-\alpha|}\right)\right)|z|^{n-1}.$$

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