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SOME INEQUALITIES FOR THE SINE INTEGRAL

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Abstract

We establish several sharp inequalities involving the function $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt$.

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Contents

1. Introduction

Let

$$\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\, dt\,,$$

be the sine integral function which plays an important role in various topics of Fourier analysis (cf. [2]). In this article we prove that the function $\operatorname{Si}(x)$ satisfies the inequalities given in the theorem below.

Theorem 1.1. *For all $x \geq 0$ and $y \geq 0$, we have*

$$0 \leq \operatorname{Si}(x) + \operatorname{Si}(y) - \operatorname{Si}(x+y) \leq 2\operatorname{Si}(\pi) - \operatorname{Si}(2\pi) = 2.285722\ldots. \tag{1.1}$$

Both bounds are sharp. We also have

$$0 \leq \operatorname{Si}(x) + \operatorname{Si}(y) \leq x + y \tag{1.2}$$

and

$$\frac{\operatorname{Si}(x)}{\operatorname{Si}(y)} \leq \frac{x}{y}, \quad \text{for} \quad x \geq y > 0\,. \tag{1.3}$$

Note that inequality (1.1) contains the sub-additive property of the function $\operatorname{Si}(x)$ and may be viewed as a two-dimensional analogue of the classical inequality

$$0 \leq \operatorname{Si}(x) \leq \operatorname{Si}(\pi) = 1.8519\ldots,$$

for all $x \geq 0$.

Inequalities (1.2) and (1.3) are also sharp because

$$\mathrm{Si}(x) = x + O(x^3), \quad \text{as} \quad x \to 0.$$

A special case of (1.2) is the following

$$0 < \frac{\mathrm{Si}(x)}{x} < 1, \quad \text{for} \quad x > 0. \tag{1.4}$$

The discrete analogue of (1.1), where the function $\mathrm{Si}(x)$ is replaced by Fejér's sums $S_n(x) = \sum_{k=1}^{n} \frac{\sin kx}{k}$, has been obtained in [1].

2. Lemmas

For the proof of inequalities (1.1) to (1.3) we need the following elementary lemmas.

Lemma 2.1. *We suppose that the function f has a continuous derivative on $[0,\infty)$ and that $f(0)=0$. If $xf'(x)\le f(x)$ for all x in $[0,\infty)$ then for $0\le t\le s$, we have $t\,f(s\,x)\le s\,f(t\,x)\le t\,x\,s\,f'(0)$ for all $x\in[0,\infty)$.*

Proof. We fix x in $[0,\infty)$ and define

$$g(t):=\frac{f(t\,x)}{t},\quad \text{for}\ \ t>0,$$

and $g(0)=x\,f'(0)$. Differentiating with respect to t we obtain

$$t^2\,g'(t)=tx\,f'(tx)-f(tx).$$

It follows from this that g is decreasing on $[0,\infty)$ therefore for $0\le t\le s$, we get $g(s)\le g(t)\le g(0)$, which completes the proof of Lemma 2.1. □

Lemma 2.2. *For all $x>0$ we have*

(2.1) $$\frac{d}{dx}\left\{\frac{1}{x}\mathrm{Si}(x)\right\}<0.$$

Proof. It is clear that (2.1) is equivalent to

(2.2) $$\mathrm{Si}(x)-\sin x>0,\quad x>0.$$

The function $\mathrm{Si}(x)$ attains its absolute minimum on $[\pi, \infty)$ at $x = 2\pi$ and $\mathrm{Si}(2\pi) = 1.4181\ldots$. Thus we have to prove (2.2) only for $0 < x < \pi$. The function $\mathrm{Si}(x)$ is strictly increasing on this interval and since $\mathrm{Si}(\pi/2) = 1.37\ldots$, it remains to show that (2.2) is valid for $0 < x < \pi/2$.

Let $h(x) := \mathrm{Si}(x) - \sin(x)$. This function is strictly increasing on $[0, \pi/2)$ because the inequality $h'(x) > 0$ is equivalent to $x \cot x < 1$ which is clearly true for this range of x and therefore the proof of (2.2) is complete. $\square$

Notice that (2.1) implies (1.4).

3. Proof of Theorem 1.1

It follows from Lemma 2.2 that the function $f(x) = \mathrm{Si}(x)$ satisfies the conditions of Lemma 2.1. Obviously $f'(0) = 1$. Therefore, for $0 \le t \le s$, we have

$$t\ \mathrm{Si}(s\,z) \le s\ \mathrm{Si}(t\,z) \le t\,s\,z, \quad \text{for all } \ z \ge 0. \tag{3.1}$$

For $x > 0,\ y > 0$, setting $z = x + y$, $t = \frac{x}{x+y}$, $s = 1$ in this inequality we obtain

$$\frac{x}{x+y}\ \mathrm{Si}(x+y) \le \mathrm{Si}(x) \le x,$$

and similarly for $z = x + y$, $t = \frac{y}{x+y}$, $s = 1$ we have

$$\frac{y}{x+y}\ \mathrm{Si}(x+y) \le \mathrm{Si}(y) \le y\,.$$

From these inequalities we conclude (1.2) and the first inequality of (1.1). Inequality (1.3) follows easily from (3.1) setting $z = 1,\ t = y,\ s = x$.

In order to prove the second inequality in (1.1) we distinguish the following cases:

a) $x + y \ge \pi$ and

b) $0 < x + y < \pi$.

In the case a) we recall that the function $\mathrm{Si}(x)$ attains its absolute maximum on $[0, \infty)$ at $x = \pi$ while its absolute minimum on $[\pi,\ \infty)$ is attained at $x = 2\pi$. Hence in this case we have

$$\mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x+y) \le 2\ \mathrm{Si}(\pi) - \mathrm{Si}(2\,\pi) = 2.285722\ldots.$$

In the case b) we consider the following subcases:

b1) $0 < x + y \leq \pi/4$,

b2) $\pi/4 < x + y \leq \pi/2$,

b3) $\pi/2 < x + y < 3\,\pi/4$ and

b4) $3\,\pi/4 < x + y < \pi$,

keeping in mind that the function $\mathrm{Si}(x)$ is strictly increasing on $[0,\, \pi]$.

In the case b1) we have

$$\mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x + y) \leq 2\ \mathrm{Si}\left(\frac{\pi}{4}\right) = 1.5179\ldots\,,$$

in the case b2) we have

$$\mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x + y) \leq 2\ \mathrm{Si}\left(\frac{\pi}{2}\right) - \mathrm{Si}\left(\frac{\pi}{4}\right) = 1.9825\ldots\,,$$

in the case b3) we have

$$\mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x + y) \leq 2\ \mathrm{Si}\left(\frac{3\,\pi}{4}\right) - \mathrm{Si}\left(\frac{\pi}{2}\right) = 2.10873\ldots$$

and finally in the case b4) we have

$$\mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x + y) \leq 2\ \mathrm{Si}(\pi) - \mathrm{Si}\left(\frac{3\,\pi}{4}\right) = 1.96412\ldots\,.$$

The numerical values of the function $\mathrm{Si}(x)$ have been calculated using Maple 8.

The proof of Theorem 1.1 is now complete. □

Remark 1. *Alternately, one can prove the inequalities in (1.1) using standard techniques from multivariate calculus. Indeed, let*

$$K(x,\,y) := \mathrm{Si}(x) + \mathrm{Si}(y) - \mathrm{Si}(x+y)\,.$$

We first observe that

$$K(0,0)=0,\quad K(x,0)=0,\quad K(0,y)=0.$$

Next, we assume that $x>0$, $y>0$. The system of equations

$$\frac{\partial}{\partial x}K(x,y)=0,\quad \frac{\partial}{\partial y}K(x,\,y)=0\,,$$

has as solutions the lattice points

$$(x,y)=(\mu\,\pi,\,\nu\,\pi),\quad \mu,\nu\in\mathbb{N}\,,$$

and this follows from the properties of the function $\sin x/x$. Using the Hessian matrix test we conclude that

1) *When μ is even and ν is odd* or *μ is odd and ν is even, the points $(\mu\,\pi,\,\nu\,\pi)$ are saddle points.*

2) *When μ is odd and ν is odd the function $K(x,\,y)$ has a local maximum at $(\mu\,\pi,\,\nu\,\pi)$.*

3) *When μ is even and ν is even the Hessian matrix test gives no information about the nature of the points $(\mu\,\pi,\,\nu\,\pi)$.*

We deal with the case 3) separately.

It is easy to see that

$$K(x,\,y) = \int_0^x \left(\frac{\sin t}{t} - \frac{\sin(t+y)}{t+y} \right) dt\,,$$

therefore, for $m,\ n = 1, 2, 3\ldots$*, we have*

$$K(2m\pi,\,2n\pi) = \int_0^{2m\pi} \left(\frac{1}{t} - \frac{1}{t+2n\pi} \right) \sin t\,dt.$$

It follows from this that

$$0 < K(2m\pi,\,2n\pi) < \int_0^{\pi} \left(\frac{1}{t} - \frac{1}{t+2n\pi} \right) \sin t\,dt < \mathrm{Si}(\pi) = 1.8519\ldots$$

Next in the case 2) we obtain for $m,\ n = 0, 1, 2\ldots$*,*

$$\begin{aligned} K\big((2m+1)\pi,\,(2n+1)\pi\big) &= \int_0^{(2m+1)\pi} \left(\frac{1}{t} + \frac{1}{t+(2n+1)\pi} \right) \sin t\,dt \\ &\le \int_0^{\pi} \left(\frac{1}{t} + \frac{1}{t+(2n+1)\pi} \right) \sin t\,dt \\ &\le \int_0^{\pi} \left(\frac{1}{t} + \frac{1}{t+\pi} \right) \sin t\,dt \\ &= 2\,\mathrm{Si}(\pi) - \mathrm{Si}(2\,\pi) = 2.285722\ldots \end{aligned}$$

This yields (1.1).

Remark 2. *Using Lemma 2.1, one can prove more general inequalities involving the function* $\mathrm{Si}(x)$*. Indeed, for the function* $f(x) = (\mathrm{Si}(x))^{\alpha}\, x^{\beta}$ *the condition* $x\, f'(x) \le f(x)$ *is equivalent to*

$$(1-\beta)\mathrm{Si}(x) - \alpha\, \sin x > 0, \quad x > 0\,. \tag{3.2}$$

This inequality is valid precisely when $\alpha + \beta \le 1$ *and* $\alpha \ge 0$*. To see this, suppose first that (3.2) holds. Dividing by* $\mathrm{Si}(x)$ *and letting* $x \to 0$ *we obtain the first condition. From (3.2) when* $\alpha + \beta \to 1$ *we get* $\alpha \ge 0$*, taking into account (2.2). Conversely, when* $\alpha + \beta \le 1$ *and* $\alpha \ge 0$*, inequality (3.2) follows from (2.2). Thus we obtain analogous results to inequalities (1.2), (1.3) and to the first inequality in (1.1) for the function* $f(x) = (\mathrm{Si}(x))^{\alpha}\, x^{\beta}$*.*

Remark 3. *Several other sharp inequalities of the type considered in this paper may be obtained using an appropriate function* $f(x)$*, which satisfies the conditions of Lemma 2.1.*

References

[1] H. ALZER AND S. KOUMANDOS, Sub- and superadditive properties of Fejér's sine polynomial, submitted.

[2] A. ZYGMUND, *Trigonometric Series*, 3rd ed., Cambridge University Press, 2002.