

Journal of Inequalities in Pure and Applied Mathematics

SOME INEQUALITIES FOR THE SINE INTEGRAL

STAMATIS KOUMANDOS

Department of Mathematics and Statistics
The University of Cyprus
P.O. Box 20537
1678 Nicosia, CYPRUS.

EMail: skoumand@ucy.ac.cy

©2000 Victoria University
ISSN (electronic): 1443-5756
251-04



volume 6, issue 1, article 25,
2005.

*Received 21 December, 2004;
accepted 25 February, 2005.*

Communicated by: H. Gauchman

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Abstract

We establish several sharp inequalities involving the function $\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt$.

2000 Mathematics Subject Classification: 26D15, 26D05, 33B10.

Key words: Sine integral function, Positive trigonometric integrals, Sharp inequalities.

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1. Introduction

Let

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt,$$

be the sine integral function which plays an important role in various topics of Fourier analysis (cf. [2]). In this article we prove that the function $\text{Si}(x)$ satisfies the inequalities given in the theorem below.

Theorem 1.1. *For all $x \geq 0$ and $y \geq 0$, we have*

$$(1.1) \quad 0 \leq \text{Si}(x) + \text{Si}(y) - \text{Si}(x + y) \leq 2 \text{Si}(\pi) - \text{Si}(2\pi) = 2.285722 \dots$$

Both bounds are sharp. We also have

$$(1.2) \quad 0 \leq \text{Si}(x) + \text{Si}(y) \leq x + y$$

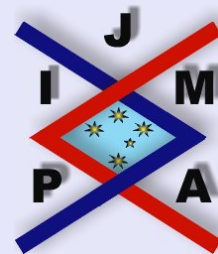
and

$$(1.3) \quad \frac{\text{Si}(x)}{\text{Si}(y)} \leq \frac{x}{y}, \text{ for } x \geq y > 0.$$

Note that inequality (1.1) contains the sub-additive property of the function $\text{Si}(x)$ and may be viewed as a two-dimensional analogue of the classical inequality

$$0 \leq \text{Si}(x) \leq \text{Si}(\pi) = 1.8519 \dots,$$

for all $x \geq 0$.



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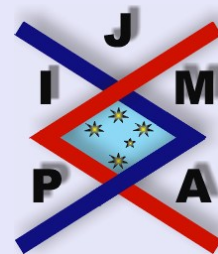
Inequalities (1.2) and (1.3) are also sharp because

$$\text{Si}(x) = x + O(x^3), \text{ as } x \rightarrow 0.$$

A special case of (1.2) is the following

$$(1.4) \quad 0 < \frac{\text{Si}(x)}{x} < 1, \text{ for } x > 0.$$

The discrete analogue of (1.1), where the function $\text{Si}(x)$ is replaced by Fejér's sums $S_n(x) = \sum_{k=1}^n \frac{\sin kx}{k}$, has been obtained in [1].



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2. Lemmas

For the proof of inequalities (1.1) to (1.3) we need the following elementary lemmas.

Lemma 2.1. *We suppose that the function f has a continuous derivative on $[0, \infty)$ and that $f(0) = 0$. If $xf'(x) \leq f(x)$ for all x in $[0, \infty)$ then for $0 \leq t \leq s$, we have $tf(sx) \leq sf(tx) \leq txsf'(0)$ for all $x \in [0, \infty)$.*

Proof. We fix x in $[0, \infty)$ and define

$$g(t) := \frac{f(tx)}{t}, \text{ for } t > 0,$$

and $g(0) = xf'(0)$. Differentiating with respect to t we obtain

$$t^2 g'(t) = txf'(tx) - f(tx).$$

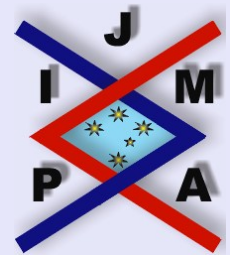
It follows from this that g is decreasing on $[0, \infty)$ therefore for $0 \leq t \leq s$, we get $g(s) \leq g(t) \leq g(0)$, which completes the proof of Lemma 2.1. \square

Lemma 2.2. *For all $x > 0$ we have*

$$(2.1) \quad \frac{d}{dx} \left\{ \frac{1}{x} \text{Si}(x) \right\} < 0.$$

Proof. It is clear that (2.1) is equivalent to

$$(2.2) \quad \text{Si}(x) - \sin x > 0, \quad x > 0.$$



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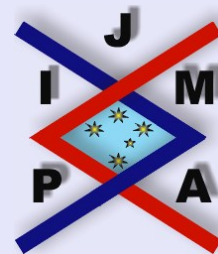
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The function $\text{Si}(x)$ attains its absolute minimum on $[\pi, \infty)$ at $x = 2\pi$ and $\text{Si}(2\pi) = 1.4181\dots$. Thus we have to prove (2.2) only for $0 < x < \pi$. The function $\text{Si}(x)$ is strictly increasing on this interval and since $\text{Si}(\pi/2) = 1.37\dots$, it remains to show that (2.2) is valid for $0 < x < \pi/2$.

Let $h(x) := \text{Si}(x) - \sin(x)$. This function is strictly increasing on $[0, \pi/2)$ because the inequality $h'(x) > 0$ is equivalent to $x \cot x < 1$ which is clearly true for this range of x and therefore the proof of (2.2) is complete. \square

Notice that (2.1) implies (1.4).



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3. Proof of Theorem 1.1

It follows from Lemma 2.2 that the function $f(x) = \text{Si}(x)$ satisfies the conditions of Lemma 2.1. Obviously $f'(0) = 1$. Therefore, for $0 \leq t \leq s$, we have

$$(3.1) \quad t \text{Si}(sz) \leq s \text{Si}(tz) \leq t s z, \quad \text{for all } z \geq 0.$$

For $x > 0$, $y > 0$, setting $z = x + y$, $t = \frac{x}{x+y}$, $s = 1$ in this inequality we obtain

$$\frac{x}{x+y} \text{Si}(x+y) \leq \text{Si}(x) \leq x,$$

and similarly for $z = x + y$, $t = \frac{y}{x+y}$, $s = 1$ we have

$$\frac{y}{x+y} \text{Si}(x+y) \leq \text{Si}(y) \leq y.$$

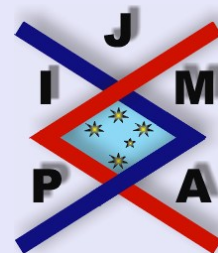
From these inequalities we conclude (1.2) and the first inequality of (1.1). Inequality (1.3) follows easily from (3.1) setting $z = 1$, $t = y$, $s = x$.

In order to prove the second inequality in (1.1) we distinguish the following cases:

- a) $x + y \geq \pi$ and
- b) $0 < x + y < \pi$.

In the case a) we recall that the function $\text{Si}(x)$ attains its absolute maximum on $[0, \infty)$ at $x = \pi$ while its absolute minimum on $[\pi, \infty)$ is attained at $x = 2\pi$. Hence in this case we have

$$\text{Si}(x) + \text{Si}(y) - \text{Si}(x+y) \leq 2 \text{Si}(\pi) - \text{Si}(2\pi) = 2.285722 \dots$$



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In the case b) we consider the following subcases:

b1) $0 < x + y \leq \pi/4,$

b2) $\pi/4 < x + y \leq \pi/2,$

b3) $\pi/2 < x + y < 3\pi/4$ and

b4) $3\pi/4 < x + y < \pi,$

keeping in mind that the function $\text{Si}(x)$ is strictly increasing on $[0, \pi]$.

In the case b1) we have

$$\text{Si}(x) + \text{Si}(y) - \text{Si}(x + y) \leq 2 \text{Si}\left(\frac{\pi}{4}\right) = 1.5179\dots,$$

in the case b2) we have

$$\text{Si}(x) + \text{Si}(y) - \text{Si}(x + y) \leq 2 \text{Si}\left(\frac{\pi}{2}\right) - \text{Si}\left(\frac{\pi}{4}\right) = 1.9825\dots,$$

in the case b3) we have

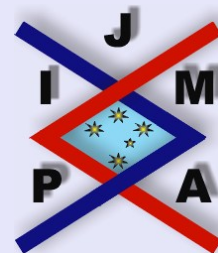
$$\text{Si}(x) + \text{Si}(y) - \text{Si}(x + y) \leq 2 \text{Si}\left(\frac{3\pi}{4}\right) - \text{Si}\left(\frac{\pi}{2}\right) = 2.10873\dots$$

and finally in the case b4) we have

$$\text{Si}(x) + \text{Si}(y) - \text{Si}(x + y) \leq 2 \text{Si}(\pi) - \text{Si}\left(\frac{3\pi}{4}\right) = 1.96412\dots$$

The numerical values of the function $\text{Si}(x)$ have been calculated using Maple 8.

The proof of Theorem 1.1 is now complete. \square



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Remark 1. Alternately, one can prove the inequalities in (1.1) using standard techniques from multivariate calculus. Indeed, let

$$K(x, y) := \text{Si}(x) + \text{Si}(y) - \text{Si}(x + y).$$

We first observe that

$$K(0, 0) = 0, \quad K(x, 0) = 0, \quad K(0, y) = 0.$$

Next, we assume that $x > 0, y > 0$. The system of equations

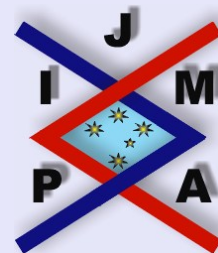
$$\frac{\partial}{\partial x} K(x, y) = 0, \quad \frac{\partial}{\partial y} K(x, y) = 0,$$

has as solutions the lattice points

$$(x, y) = (\mu \pi, \nu \pi), \quad \mu, \nu \in \mathbb{N},$$

and this follows from the properties of the function $\sin x/x$. Using the Hessian matrix test we conclude that

- 1) When μ is even and ν is odd or μ is odd and ν is even, the points $(\mu \pi, \nu \pi)$ are saddle points.
- 2) When μ is odd and ν is odd the function $K(x, y)$ has a local maximum at $(\mu \pi, \nu \pi)$.
- 3) When μ is even and ν is even the Hessian matrix test gives no information about the nature of the points $(\mu \pi, \nu \pi)$.



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We deal with the case 3) separately.

It is easy to see that

$$K(x, y) = \int_0^x \left(\frac{\sin t}{t} - \frac{\sin(t+y)}{t+y} \right) dt,$$

therefore, for $m, n = 1, 2, 3 \dots$, we have

$$K(2m\pi, 2n\pi) = \int_0^{2m\pi} \left(\frac{1}{t} - \frac{1}{t+2n\pi} \right) \sin t dt.$$

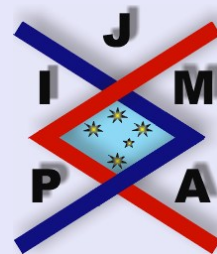
It follows from this that

$$0 < K(2m\pi, 2n\pi) < \int_0^\pi \left(\frac{1}{t} - \frac{1}{t+2n\pi} \right) \sin t dt < \text{Si}(\pi) = 1.8519 \dots$$

Next in the case 2) we obtain for $m, n = 0, 1, 2 \dots$,

$$\begin{aligned} K((2m+1)\pi, (2n+1)\pi) &= \int_0^{(2m+1)\pi} \left(\frac{1}{t} + \frac{1}{t+(2n+1)\pi} \right) \sin t dt \\ &\leq \int_0^\pi \left(\frac{1}{t} + \frac{1}{t+(2n+1)\pi} \right) \sin t dt \\ &\leq \int_0^\pi \left(\frac{1}{t} + \frac{1}{t+\pi} \right) \sin t dt \\ &= 2 \text{Si}(\pi) - \text{Si}(2\pi) = 2.285722 \dots \end{aligned}$$

This yields (1.1).



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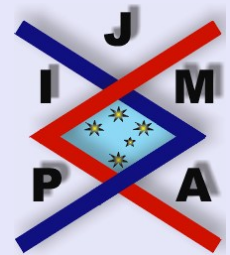
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Remark 2. Using Lemma 2.1, one can prove more general inequalities involving the function $\text{Si}(x)$. Indeed, for the function $f(x) = (\text{Si}(x))^\alpha x^\beta$ the condition $x f'(x) \leq f(x)$ is equivalent to

$$(3.2) \quad (1 - \beta)\text{Si}(x) - \alpha \sin x > 0, \quad x > 0.$$

This inequality is valid precisely when $\alpha + \beta \leq 1$ and $\alpha \geq 0$. To see this, suppose first that (3.2) holds. Dividing by $\text{Si}(x)$ and letting $x \rightarrow 0$ we obtain the first condition. From (3.2) when $\alpha + \beta \rightarrow 1$ we get $\alpha \geq 0$, taking into account (2.2). Conversely, when $\alpha + \beta \leq 1$ and $\alpha \geq 0$, inequality (3.2) follows from (2.2). Thus we obtain analogous results to inequalities (1.2), (1.3) and to the first inequality in (1.1) for the function $f(x) = (\text{Si}(x))^\alpha x^\beta$.

Remark 3. Several other sharp inequalities of the type considered in this paper may be obtained using an appropriate function $f(x)$, which satisfies the conditions of Lemma 2.1.



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References

- [1] H. ALZER AND S. KOUMANDOS, Sub- and superadditive properties of Fejér's sine polynomial, submitted.
- [2] A. ZYGMUND, *Trigonometric Series*, 3rd ed., Cambridge University Press, 2002.



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