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ON GLOBAL APPROXIMATION PROPERTIES OF ABSTRACT INTEGRAL OPERATORS IN ORLICZ SPACES AND APPLICATIONS

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ABSTRACT. In this paper we study approximation properties for the class of general integral operators of the form

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_{H_w}(t) \; s \in G, \; w > 0$$

where G is a locally compact Hausdorff topological space, $(H_w)_{w>0}$ is a net of closed subsets of G with suitable properties and, for every $w > 0$, μ_{H_w} is a regular measure on H_w. We give pointwise, uniform and modular convergence theorems in abstract modular spaces and we apply the results to some kinds of discrete operators including the sampling type series.

Key words and phrases: Modular approximation, nonlinear integral operators, regular families, singularity.

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1. INTRODUCTION

The aim of this paper is to give a general and unifying treatment of convergence properties in function spaces for families of integral or discrete operators, including the classical linear or nonlinear integral operators of convolution type acting on function spaces defined on topological groups, Urysohn type operators, discrete operators of sampling type, or the classical Bernstein and Szász-Mirak'jan operators etc. Our objective is to find a common root for the convergence properties of all the above operators.

In [4] we have studied approximation properties for families of nonlinear integral operators of the form

$$(T_w f)(s) = \int_{H_w} K_w(s,t,f(t))d\mu_{H_w}(t), \; s \in G, \; w > 0,$$

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where G is a locally compact Hausdorff topological space, $(H_w)_{w>0}$ is a net of closed subsets of G such that $\overline{\bigcup_{w>0} H_w} = G$ and, for every $w > 0$, μ_{H_w} is a regular and σ-*finite* measure on H_w. For these operators we have obtained some local convergence theorems in the general setting of modular spaces. Locality represents a restriction of applicability of the theory and so in this paper we furnish global approximation results by using a notion of uniform modular integrability for families of functions introduced in [5] for Urysohn type operators and suitably modified according to our general frame. We will work in the setting of Orlicz spaces, but using some natural generalization of the uniform integrability, some technical tool and an approach similar to [4], we can obtain corresponding results also in abstract modular spaces. As a consequence we obtain results in L^p spaces, when the generating function of the Orlicz space is given by $\varphi(u) = |u|^p$.

By specializing G and H_w, we obtain several classes of operators. For example, if $G = H_w$, and $\mu_w = \mu_G$, for every $w > 0$, μ_G being a fixed $\sigma-$finite and regular measure on G, we obtain a class of Urysohn type operators and so, in turn, all the classical families of linear or nonlinear integral operators of convolution type (when G is a topological group and μ_G is its Haar measure). Our main theorems (Theorems 3.4 and 3.6) imply the well-known approximation results in Orlicz spaces (or, in particular, in L^p-spaces), see [13], [24]).

Here we consider, as examples, very interesting families of operators of the discrete type (see Section 4), in order to explain the nature of the general conditions which we used. In particular, we apply these results to linear discrete operators of the form ([26], [14], [15], [16], [7], [22] and [6])

$$(S_w f)(s) = \sum_{n\in\mathbb{Z}} K(ws - n) f\left(\frac{n}{w}\right),\ s \in \mathbb{R},\ w > 0$$

and

$$(\widetilde{S}_w f)(s) = \sum_{n\in\mathbb{Z}} G_{n,w}(s) f\left(\frac{t_n}{w}\right)\ s \in \mathbb{R},\ w > 0,$$

where $(t_n)_{n\in\mathbb{N}}$ is a separate sequence of real numbers (i.e. $|t_n - t_{n-1}| \geq \delta$ for a fixed $\delta > 0$), (see [21], [17], [3]) and to their nonlinear versions (see e.g. [8], [27], [23], [6], [3], [4] and [1]). In these instances, our general convergence theorems imply modular convergence in Orlicz spaces for the generalized sampling operators (in their linear or nonlinear forms), when the function f satisfies some generalized concepts of bounded variation. As far we know, this is a new result about generalized sampling operators (also in L^p-case). Finally, we also discuss another well-known discrete operator, namely the Szász-Mirak'jan operator and we obtain, as an application, a convergence result in Orlicz space.

2. NOTATIONS AND DEFINITIONS

Let G be a locally compact Hausdorff topological space, provided with its family of Borel sets $\mathcal{B}$ of G. Let μ_G be a regular and $\sigma-$finite measure defined on $\mathcal{B}$. We will assume that the topology of G is uniformizable, i.e. there is a uniform structure $\mathcal{U} \subset G \times G$ which generates the topology of G (see [28]). For every $U \in \mathcal{U}$, we put $U_s = \{t \in G : (s,t) \in U\}$. By local compactness, we will assume that for every $s \in G$, the base $\{U_s : U \in \mathcal{U}\}$ contains compact sets. We will denote by $X(G)$ the space of all real-valued measurable functions $f : G \to \mathbb{R}$ provided with equality a.e., by $C(G)$ the subspace of all bounded and continuous functions and by $C_c(G)$ the subspace of all continuous functions with compact support.

Let us recall that a function $f : G \to \mathbb{R}$ is uniformly continuous on G if for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|f(t) - f(s)| < \varepsilon$ for every $s \in G, t \in U_s$.

Let now $(H_w)_{w>0}$ be a net of nonempty closed subsets of G such that $G = \overline{\cup_{w>0} H_w}$.

For every $w > 0$ we will denote by μ_w a regular and σ-finite measure on H_w, defined on the Borel σ-algebra generated by the family $\{A \cap H_w : \text{A open subset of } G\}$.

Let $\mathcal{L}$ be the class of all the families of globally measurable functions $\mathbb{L} = (L_w)_{w>0}$, $L_w : G \times H_w \to \mathbb{R}_0^+$ such that for every $w > 0$ the sections $L_w(\cdot, t)$ and $L_w(s, \cdot)$ belong to $L^1(G)$ for every $t \in H_w$ and to $L^1(H_w)$ for every $s \in G$ respectively.

Let Ψ be the class of all functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that ψ is a continuous (nondecreasing) function and $\psi(0) = 0,\ \psi(u) > 0$ for $u > 0$. Let $\Xi = (\psi_w)_{w>0} \subset \Psi$ be a family of functions such that the following two assumptions hold

1. $(\psi_w)_{w>0}$ is equicontinuous at $u = 0$,
2. for every $u \geq 0$ the net $(\psi_w(u))_{w>0}$ is bounded.

We denote by $\mathcal{K}_\Xi$ the class of all families of functions $\mathbb{K} = (K_w)_{w>0}$, where $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}$, such that the following conditions hold

(K.1) for any $w > 0$, $K_w(\cdot, \cdot, u)$ is measurable on $G \times H_w$ for every $u \in \mathbb{R}$ and $K_w(s, t, 0) = 0$, for every $(s, t) \in G \times H_w$.

(K.2) $\mathbb{K} = (K_w)_{w>0}$ is $(\mathbb{L}, \Xi)-$Lipschitz i.e. there are $\mathbb{L} = (L_w)_{w>0} \in \mathcal{L}$ and a constant $D > 0$ such that

$$0 < \beta_w(s) := \int_{H_w} L_w(s, t) d\mu_w(t) \leq D$$

for all $s \in G, w > 0$ and

$$|K_w(s, t, u) - K_w(s, t, v)| \leq L_w(s, t)\psi_w(|u - v|)$$

for every $s \in G$, $t \in H_w$ and $u, v \in \mathbb{R}$.

For a given $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ we will take into consideration the following family of nonlinear integral operators $\mathbf{T} = (T_w)_{w>0}$ given by

$$(T_w f)(s) = \int_{H_w} K_w(s, t, f(t)) d\mu_w(t)\ s \in G,\ w > 0$$

where $f \in \operatorname{Dom} \mathbf{T} = \cap_{w>0} \operatorname{Dom} T_w$; here $\operatorname{Dom} T_w$ is the subset of $X(G)$ on which $T_w f$ is well defined as a μ_G-measurable function of $s \in G$.

We will say that $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ is *singular* if the following assumptions hold

1) There is a net $(\zeta_w)_{w>0}$ of positive real numbers such that for every $w > 0$ and $t \in H_w$ we have

$$\int_G L_w(s, t) d\mu_G(s) \leq \zeta_w \leq D.$$

2) For every $s \in G$ and for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{H_w \setminus U_s} L_w(s, t) d\mu_w(t) = 0.$$

3) For every $s \in G$ and for every $u \in \mathbb{R}$ we have

$$\lim_{w\to+\infty} \int_{H_w} K_w(s, t, u) d\mu_w(t) = u.$$

We will say that $\mathbb{K}$ is *uniformly singular* if conditions 2) and 3) are replaced by the following ones

2') for every $U \in \mathcal{U}$ we have

$$\lim_{w\to+\infty} \int_{H_w \setminus U_s} L_w(s, t) d\mu_w(t) = 0$$

uniformly with respect to $s \in G$,

3') we have

$$\lim_{w\to+\infty}\int_{H_w} K_w(s,t,u)d\mu_w(t) = u,$$

uniformly with respect to $s \in G$ and $u \in C$, where C is any compact subset of $\mathbb{R}\setminus\{0\}$.

For the above concepts see [4].

We have the following

Theorem 2.1. *Let* $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$. *Then* $L^\infty(G) \subset \mathrm{Dom}\,\mathbf{T}$ *and for every* $w > 0$ $T_w f \in L^\infty(G)$, *whenever* $f \in L^\infty(G)$.

Proof. We obtain easily the measurability of $K_w(s,\cdot,f(\cdot))$ and $T_w f$, with $f \in L^\infty(G)$. Moreover

$$|(T_w f)(s)| \leq \int_{H_w} L_w(s,t)\psi_w(|f(t)|)d\mu_w(t) \leq \psi_w(\|f\|_\infty)D \leq M'D,$$

being $M' = \sup_{w>0}(\psi_w(\|f\|_\infty))$ and so the assertion follows. □

As to the pointwise convergence, in [4] we have proved the following result

Theorem 2.2. *Let* $f \in L^\infty(G)$ *and* $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ *be singular. Then* $T_w f \to f$ *pointwise at every continuity point of* f. *Moreover if* $\mathbb{K}$ *is uniformly singular, then* $\|T_w f - f\|_\infty \to 0$ *as* $w \to +\infty$, *whenever* $f \in C(G)$ *is uniformly continuous.*

3. CONVERGENCE IN ORLICZ SPACES

Let Φ be the class of all functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

i) φ is continuous and non decreasing,

ii) $\varphi(0) = 0,\ \varphi(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty}\varphi(u) = +\infty$.

Moreover we denote by $\widetilde{\Phi}$ the subspace of Φ whose elements are convex functions.

For $\varphi \in \Phi$, we define the functional

$$\varrho_G^\varphi(f) = \int_G \varphi(|f(s)|)d\mu_G(s)$$

for every $f \in X(G)$.

As it is well known, ϱ_G^φ is a modular on $X(G)$ and the subspace

$$L^\varphi(G) = \{f \in X(G) : \varrho_G^\varphi(\lambda f) < +\infty \text{ for some } \lambda > 0\}$$

is the Orlicz space generated by φ. If $\varphi \in \widetilde{\Phi}$, then ϱ_G^φ is a convex modular. The subspace of $L^\varphi(G)$, defined by

$$E^\varphi(G) = \{f \in X(G) : \varrho_G^\varphi(\lambda f) < +\infty \text{ for every } \lambda > 0\},$$

is called the space of finite elements of $L^\varphi(G)$. For example, every bounded function with compact support belongs to $E^\varphi(G)$. For these concepts see [24].

Moreover, given an arbitrary $\eta \in \Phi$, we will denote by $\varrho_{H_w}^\eta$ the modular

$$\varrho_{H_w}^\eta(f) = \int_{H_w} \eta(|f(t)|)d\mu_w(t)$$

for any $f \in X(G)$.

Here by $L^\eta(H_w)$ we denote the space of all functions $f \in X(G)$ such that the restriction $f_{|H_w}$ is an element of the Orlicz space generated by the modular $\varrho_{H_w}^\eta$.

In order to establish a convergence result in Orlicz spaces, we consider the following notion of convergence (see [24]). We say that a sequence $(f_w)_{w>0} \subset L^\varphi(G)$ is modularly convergent to $f \in L^\varphi(G)$ if there is $\lambda > 0$ such that

$$\lim_{w\to+\infty} \varrho_G^\varphi[\lambda(f_w - f)] = 0.$$

This notion of convergence induces a topology on $L^\varphi(G)$, called modular topology.

In the following we will need a link between the modular $\varrho^\varphi_{H_w}$ and the family of functions $\Xi = (\psi_w)_{w>0}$ used in the Lipschitz condition of $(K_w)_{w>0}$. Given a function $\eta \in \Phi$, considering the modular $\varrho^\eta_{H_w}$, we will say that the triple $(\varrho^\varphi_{H_w}, \psi_w, \varrho^\eta_{H_w})$ is properly directed if the following condition holds: there exists a net $(c_w)_{w>0}$, with $c_w \to 0$ as $w \to +\infty$, for which for every $\lambda \in]0,1[$ there exists $C_\lambda \in]0,1[$ with

$$\int_{H_w} \varphi(C_\lambda \psi_w(|f(t)|)) d\mu_w(t) \leq \int_{H_w} \eta(\lambda|f(t)|) d\mu_w(t) + c_w \tag{3.1}$$

for every function $f \in X(G)$.

We will need the following definition: given

$$(L_w(\cdot, t))_{t\in H_w, w>0},$$

we will say that this family satisfies property $(*)$, if for every compact $C \subset G$ there exist a compact subset $B \subset G$, $M > 0$ and $\beta > 0$ such that

$$\int_{G\setminus B} L_w(s,t) d\mu_G(s) \leq \frac{M}{w^\beta} \tag{3.2}$$

for every $t \in C \cap H_w$ and sufficiently large $w > 0$.

3.1. Examples.

1. As a first example, let $G = H_w = \mathbb{R}$, for every $w > 0$, provided with the Lebesgue measure. Let us consider an approximate identity (see [13]) $(\widetilde{K}_w)_{w>0}$ on $G = \mathbb{R}$, such that for every $\delta > 0$

   $$\lim_{w\to+\infty} w^\beta \int_{|t|>\delta} |\widetilde{K}_w(s)| ds = M > 0,$$

   for some $\beta > 0$. Then we can prove that the family of kernels $(K_w)_{w>0}$, with $K_w : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

   $$K_w(s,t,u) = \widetilde{K}_w(s-t)u,$$

   satisfies the singularity assumptions and the family $L_w(s,t) = |\widetilde{K}_w(s-t)|$ satisfies property $(*)$. For example the above holds for every family $(\widetilde{K}_w)_{w>0}$ of functions $\widetilde{K}_w$ with supports contained in a fixed interval of the form $[-a,a]$.
2. Let $G = H_w = \mathbb{R}$ for every $w > 0$, provided with Lebesgue measure. Let us consider the family of functions

   $$L_w(s,t) = \frac{2w}{\pi} \cdot \frac{e^{w(s+t)}}{e^{2ws} + e^{2wt}},$$

   for $s, t \in \mathbb{R}$ and $w \geq 1$. Then for every $t \in \mathbb{R}$ and $r > 0$ we have

   $$\int_{\mathbb{R}\setminus[t-r,t+r]} L_w(s,t) ds = 1 + \frac{2}{\pi}(\arctan e^{-wr} - \arctan e^{wr}).$$

   Now it is easy to show that (3.2) holds for every $\beta > 0$. This is an example in non-convolution case (see [4]).

3. More interesting cases, which involve discrete operators, will be given in Section 4. These examples can be easily modified in order to get conditions for which the assumption $(*)$ holds.

Let $\eta \in \Phi$ be fixed. For a given constant $P > 0$ and a net $(\gamma_w)_{w>0}$ of positive numbers, we denote by $\Upsilon_{H_w}(P, (\gamma_w)_{w>0})$ the class

$$\Upsilon_{H_w}(P, (\gamma_w)_{w>0}) = \{f \in L^\eta(G) : \limsup_{w\to+\infty} \gamma_w \varrho^\eta_{H_w}(\lambda f) \le P\varrho^\eta_G(\lambda f) \text{ for every } \lambda > 0\}.$$

Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular and let $(\zeta_w)_{w>0}$ be the net in assumption 1) of singularity.

In the following we will work with the class $\Upsilon_{H_w}(P, (\zeta_w)_{w>0})$ and we will assume that $\Upsilon_{H_w}(P, (\zeta_w)_{w>0})$ contains a nonempty subspace $\mathcal{F} \subset C_c(G)$. For a fixed modular functional ϱ^η_G on $X(G)$, we will denote by $\overline{\mathcal{F}}_\eta$ the modular closure in $L^\eta(G)$ of $\mathcal{F}$. Thus $f \in \overline{\mathcal{F}}_\eta$ iff there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $(f_n)_{n\in\mathbb{N}}$ converges modularly to f with respect to the modular ϱ^η_G. We have the following

Theorem 3.1. *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$, $\Xi = (\psi_w)_{w>0} \subset \Psi$ and let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let $f \in \mathcal{F}$ and let C be the support of f. Let us assume that $w^\beta\zeta_w \to +\infty$ as $w \to +\infty$ and (3.1) holds. If the family $(L_w(\cdot, t))_{t\in H_w, w>0}$ satisfies property $(*)$, then there exists a constant $\alpha > 0$, independent of f, such that following properties hold*

i) *for every $\varepsilon > 0$ there is a compact set $B \subset G$ such that*

$$\varrho^\varphi_G[\alpha(T_w f)\chi_{G\setminus B}] < \varepsilon$$

for sufficiently large $w > 0$,

ii) *for every sequence $(B_k)_{k\in\mathbb{N}} \subset \mathcal{B}$, with $B_{k+1} \subset B_k$ and $\mu_G(B_k) \to 0$, we have*

$$\lim_{k\to+\infty} \varrho^\varphi_G[\alpha(T_w f)\chi_{B_k}] = 0,$$

uniformly with respect to $w > 0$.

Proof. Let $\lambda > 0$ be fixed and let C_λ be the constant in (3.1). Let $\alpha > 0$ be a constant such that $\alpha D \le C_\lambda$. Using the Jensen inequality and the Fubini-Tonelli Theorem, we have for every measurable subset $B \subset G$

$$\begin{aligned}\varrho^\varphi_G[\alpha(T_w f)\chi_{G\setminus B}] &= \int_{G\setminus B} \varphi(\alpha|(T_w f)(s)|)d\mu_G(s)\\ &\le \frac{1}{D}\int_{G\setminus B}\left[\int_{H_w\cap C} \varphi(\alpha D\psi_w(|f(t)|))L_w(s,t)d\mu_w(t)\right]d\mu_G(s)\\ &= \frac{1}{D}\int_{H_w\cap C} \varphi(\alpha D\psi_w(|f(t)|))\left[\int_{G\setminus B} L_w(s,t)d\mu_G(s)\right]d\mu_w(t).\end{aligned}$$

By property $(*)$ we can consider a compact subset $B \subset G$ such that

$$\int_{G\setminus B} L_w(s,t)d\mu_G(s) \le \frac{M}{w^\beta},$$

for every $t \in H_w \cap C$, with $\beta > 0$. So we obtain

$$\varrho^\varphi_G[\alpha(T_w f)\chi_{G\setminus B}] \le \frac{M}{Dw^\beta}\int_{H_w} \varphi(\alpha D\psi_w(|f(t)|))d\mu_w(t).$$

Using condition (3.1), we obtain

$$\begin{aligned}\varrho^{\varphi}_{G}[\alpha(T_w f)\chi_{G\setminus B}] &\leq \frac{M}{Dw^{\beta}}\left[\int_{H_w}\eta(\lambda|f(t)|)d\mu_w(t)+c_w\right]\\ &= \frac{M}{Dw^{\beta}}\int_{H_w}\eta(\lambda|f(t)|)d\mu_w(t)+\frac{Mc_w}{Dw^{\beta}}\\ &= \frac{M}{Dw^{\beta}\zeta_w}\zeta_w\int_{H_w}\eta(\lambda|f(t)|)d\mu_w(t)+\frac{Mc_w}{Dw^{\beta}}.\end{aligned}$$

Letting $w\to+\infty$, we have

$$\limsup_{w\to+\infty}\varrho^{\varphi}_{G}[\alpha(T_w f)\chi_{G\setminus B}]\leq\limsup_{w\to+\infty}\frac{M}{Dw^{\beta}\zeta_w}\zeta_w\int_{H_w}\eta(\lambda|f(t)|)d\mu_w(t)$$

and, taking into account that $f\in\Upsilon_{H_w}(P,(\zeta_w)_{w>0})$ and that every function $f\in\mathcal{F}$ belongs to $E^{\eta}(G)$, we have that

$$\limsup_{w\to+\infty}\varrho^{\varphi}_{G}[\alpha(T_w f)\chi_{G\setminus B}]=0$$

and so i) follows. Now we can prove ii). Let $(B_k)_{k\in\mathbb{N}}\subset\mathcal{B}$ be a sequence of measurable sets with $B_{k+1}\subset B_k$ and $\lim_{k\to+\infty}\mu_G(B_k)=0$. Then, considering a constant α such that $\alpha D\leq 1$, we obtain

$$\begin{aligned}\varrho^{\varphi}_{G}[\alpha(T_w f)\chi_{B_k}] &= \int_G\varphi(\alpha|(T_w f)(s)|\chi_{B_k}(s))d\mu_G(s)\\ &\leq \varphi(\psi_w(\|f\|_{\infty}))\mu_G(B_k).\end{aligned}$$

Since $(\psi_w(\|f\|_{\infty}))_{w>0}$ is bounded we have easily the assertion. □

Using Theorems 2.2 and 3.1, we obtain the following

Theorem 3.2. *Let $\varphi\in\widetilde{\Phi}$, $\eta\in\Phi$, $\Xi=(\psi_w)_{w>0}\subset\Psi$ and $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_{\Xi}$ be singular. Let us assume that $w^{\beta}\zeta_w\to+\infty$ as $w\to+\infty$ and (3.1) holds. If the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(*)$, then there exists a constant $\alpha>0$ such that*

$$\lim_{w\to+\infty}\varrho^{\varphi}_{G}[\alpha(T_w f-f)]=0,$$

for every $f\in\mathcal{F}$.

Proof. Let $\lambda>0$ be fixed and let $\alpha>0$ be such that $2\alpha D\leq C_{\lambda}$. From Theorem 2.2, $T_w f$ converges pointwise to f so by continuity of φ we have that $\varphi(\alpha|T_w f-f|)$ tends to zero pointwise too. Moreover by Theorem 3.1 the integrals

$$\int_{(\cdot)}\varphi(\alpha|(T_w f)(s)-f(s)|)d\mu_G(s)$$

are equiabsolutely continuous. Thus the assertion follows from the Vitali convergence theorem. □

In order to give a modular convergence theorem for functions $f\in\overline{\mathcal{F}}_{\varphi+\eta}(G)\cap\operatorname{Dom}\mathbf{T}$, we prove the following continuity result for the family $(T_w)_{w>0}$.

Theorem 3.3. *Let $\varphi\in\widetilde{\Phi}$, $\eta\in\Phi$ and $\Xi=(\psi_w)_{w>0}\subset\Psi$ be such that the triple $(\varrho^{\varphi}_{H_w},\psi_w,\varrho^{\eta}_{H_w})$ is properly directed. Let $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_{\Xi}$ be singular. If $f,g\in X(G)\cap\operatorname{Dom}\mathbf{T}$, such that $f-g\in\Upsilon_{H_w}(P,(\zeta_w)_{w>0})$ then, given $\lambda>0$, there exists $\alpha>0$ such that*

$$\limsup_{w\to+\infty}\varrho^{\varphi}_{G}[\alpha(T_w f-T_w g)]\leq\frac{P}{D}\varrho^{\eta}_{G}[\lambda(f-g)].$$

Proof. Let $\lambda > 0$ be fixed and let α be a positive constant such that $\alpha D \leq C_\lambda$, with C_λ being the constant in (3.1). By using the Jensen inequality and the Fubini-Tonelli theorem, we have

$$\begin{aligned}
\varrho^\varphi_G[\alpha(T_w f - T_w g)] &= \int_G \varphi(\alpha|(T_w f)(s) - (T_w g)(s)|)d\mu_G(s) \\
&\leq \int_G \varphi\left(\alpha \int_{H_w} L_w(s,t)\psi_w(|f(t) - g(t)|)d\mu_w(t)\right) d\mu_G(s) \\
&\leq \frac{1}{D}\int_{H_w} \varphi(C_\lambda \psi_w(|f(t) - g(t)|)) \left(\int_G L_w(s,t)d\mu_G(s)\right) d\mu_w(t) \\
&\leq \frac{\zeta_w}{D}\int_{H_w} \varphi(C_\lambda \psi_w(|f(t) - g(t)|))d\mu_w(t) \\
&\leq \frac{\zeta_w}{D}\int_{H_w} \eta(\lambda|f(t) - g(t)|)d\mu_w(t) + \frac{c_w\zeta_w}{D} \\
&= \frac{\zeta_w}{D}\varrho^\eta_{H_w}[\lambda(f-g)] + \frac{c_w\zeta_w}{D}.
\end{aligned}$$

Passing to the limsup, taking into account that $f - g \in \Upsilon_{H_w}(P, (\zeta_w)_{w>0})$, we obtain

$$\limsup_{w\to+\infty} \varrho^\varphi_G[\alpha(T_w f - T_w g)] \leq \frac{P}{D}\varrho^\eta_G(\lambda(f-g))$$

and so the assertion follows. □

Now we are ready to prove the main theorem of this section.

Theorem 3.4. *Let $\varphi \in \widetilde{\Phi}$, $\eta \in \Phi$ and $\Xi = (\psi_w)_{w>0} \subset \Psi$ be such that the triple $(\varrho^\varphi_{H_w}, \psi_w, \varrho^\eta_{H_w})$ is properly directed. Let $\mathbb{K} = (K_w)_{w>0} \in \mathcal{K}_\Xi$ be singular. Let us assume that $w^\beta\zeta_w \to +\infty$ as $w \to +\infty$, and the family $(L_w(\cdot,t))_{t\in H_w, w>0}$ satisfies property $(*)$. Then, for every $f \in \mathrm{Dom}\,\mathbf{T} \cap \overline{\mathcal{F}}_{\eta+\varphi}$, such that $f - \mathcal{F} \subset \Upsilon_{H_w}(P, (\zeta_w)_{w>0})$, for some $P > 0$, there exists a constant $\alpha > 0$ such that*

$$\lim_{w\to+\infty} \varrho^\varphi_G[\alpha(T_w f - f)] = 0.$$

Proof. Let $f \in \overline{\mathcal{F}}_{\eta+\varphi} \cap \mathrm{Dom}\,\mathbf{T}$, such that $f - \mathcal{F} \subset \Upsilon_{H_w}(P, (\zeta_w)_{w>0})$. Then there is a constant $\lambda' \in]0,1[$ and a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that for every $\varepsilon > 0$ there exists $\overline{n} \in \mathbb{N}$ with

$$\varrho^{\varphi+\eta}_G[\lambda'(f_n - f)] < \varepsilon$$

for every $n \geq \overline{n}$. Let us fix now $\overline{n}$ and a constant $\alpha > 0$ such that $3\alpha \leq \lambda'$, for which Theorems 3.2 and 3.3 are satisfied with 3α. Then we have

$$\varrho^\varphi_G[\alpha(T_w f - f)] \leq \varrho^\varphi_G[3\alpha(T_w f - T_w f_{\overline{n}})] + \varrho^\varphi_G[3\alpha(T_w f_{\overline{n}} - f_{\overline{n}})] + \varrho^\varphi_G[3\alpha(f_{\overline{n}} - f)].$$

By Theorem 3.2 we have

$$\lim_{w\to+\infty} \varrho^\varphi_G[3\alpha(T_w f_{\overline{n}} - f_{\overline{n}})] = 0$$

and by Theorem 3.3 we have

$$\limsup_{w\to+\infty} \varrho^\varphi_G[3\alpha(T_w f - T_w f_{\overline{n}}) \leq \frac{P}{D}\varrho^\eta_G(\lambda'(f - f_{\overline{n}})).$$

Without loss of generality we can assume that $P/D > 1$ and so we can write

$$\limsup_{w\to+\infty} \varrho^\varphi_G[\alpha(T_w f - f)] \leq \frac{P}{D}\varrho^\eta_G[\lambda'(f - f_{\overline{n}})] + \frac{P}{D}\varrho^\varphi_G[\lambda'(f - f_{\overline{n}})] < \varepsilon\frac{P}{D}.$$

The assertion follows since ε is arbitrary. □

Now we describe a different approach to the convergence problem by using a more general assumption than property $(*)$. This assumption has been used in [5] in connection with Urysohn type operators. However we have to introduce a "local" boundedness condition on the family of measures $(\mu_w)_{w>0}$. Using the previous notations, we will say that the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(**)$ if for every compact $C\subset G$ and for every $\varepsilon>0$ there exists a compact subset $B\subset G$ such that

$$\int_{G\setminus B} L_w(s,t)d\mu_G(s)<\varepsilon,$$

for every $t\in H_w\cap C$ and sufficiently large $w>0$.

We will say that the family of measures $(\mu_w)_{w>0}$ is *locally bounded* if for every compact $C\subset G$, $(\mu_w(C\cap H_w))_{w>0}$ is a bounded net, i.e. there exists a constant $R=R_C>0$ such that $\mu_w(C\cap H_w)\leq R$, for sufficiently large $w>0$.

As before, let $\mathcal{F}$ be a subspace of $C_c(G)$ such that $\mathcal{F}\subset \Upsilon_{H_w}(P,(\zeta_w)_{w>0})$ for some constant $P>0$. We have the following

Theorem 3.5. *Let $\varphi\in\widetilde{\Phi}$, $\Xi=(\psi_w)_{w>0}\subset\Psi$ and let $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_\Xi$ be singular. Let $f\in\mathcal{F}$ and let C be the support of f. If the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(**)$ and the family of measures $(\mu_w)_{w>0}$ is locally bounded, then there exists a constant $\alpha>0$, independent of f, such that following properties hold*

i) *for every $\varepsilon>0$ there is a compact set $B\subset G$ such that*

$$\varrho^\varphi_G[\alpha(T_wf)\chi_{G\setminus B}]<\varepsilon$$

for sufficiently large $w>0$,

ii) *for every sequence $(B_k)_{k\in\mathbb{N}}\subset\mathcal{B}$, with $B_{k+1}\subset B_k$ and $\mu_G(B_k)\to 0$, we have*

$$\lim_{k\to+\infty}\varrho^\varphi_G[\alpha(T_wf)\chi_{B_k}]=0,$$

uniformly with respect to $w>0$.

Proof. Let $\alpha>0$ be such that $\alpha D\leq 1$. Let $\varepsilon>0$ be fixed and let B be the compact set corresponding to the support C of f in the definition of property $(**)$. Using the same arguments as in Theorem 3.1, we have, for sufficiently large w,

$$\begin{aligned}\varrho^\varphi_G[\alpha(T_wf)\chi_{G\setminus B}] &\leq \frac{1}{D}\int_{H_w\cap C}\varphi(\alpha D\psi_w(|f(t)|))\left[\int_{G\setminus B}L_w(s,t)d\mu_G(s)\right]d\mu_w(t)\\ &\leq\frac{\varepsilon}{D}\int_{H_w\cap C}\varphi(\psi_w(|f(t)|))d\mu_w(t)\leq\frac{\varepsilon R}{D}\varphi(M'),\end{aligned}$$

where $M'=\sup_{w>0}\psi_w(\|f\|_\infty)$ and so i) follows. The proof of ii) is exactly the same as before. □

We remark that Theorem 3.5 still holds for every bounded function with compact support.

As a consequence we obtain a modular convergence result as in Theorem 3.2. Finally, using Theorem 3.3, Theorem 3.5 and the Vitali convergence Theorem, we obtain the restatement of Theorem 3.4

Theorem 3.6. *Let $\varphi\in\widetilde{\Phi}$, $\eta\in\Phi$ and $\Xi=(\psi_w)_{w>0}\subset\Psi$ be such that the triple $(\varrho^\varphi_{H_w},\psi_w,\varrho^\eta_{H_w})$ is properly directed. Let $\mathbb{K}=(K_w)_{w>0}\in\mathcal{K}_\Xi$ be singular. Let us assume that the family $(L_w(\cdot,t))_{t\in H_w,w>0}$ satisfies property $(**)$ and the family of measures $(\mu_w)_{w>0}$ is locally bounded.*

Then, for every $f \in \mathrm{Dom}\,\mathbf{T} \cap \overline{\mathcal{F}}_{\eta+\varphi}$, *such that* $f - \mathcal{F} \subset \Upsilon_{H_w}(P, (\zeta_w)_{w>0})$, *for some* $P > 0$, *there exists a constant* $\alpha > 0$ *such that*

$$\lim_{w\to+\infty} \varrho^{\varphi}_{G}[\alpha(T_w f - f)] = 0.$$

4. Applications to Discrete Operators

Let us consider now the particular case in which $G = \mathbb{R}$ and $H_w = \frac{1}{w}\mathbb{Z}$ for every $w > 0$. We provide G with the Lebesgue measure and for every $w > 0$ we put $\mu_w = \frac{1}{w}\mu_c$, where μ_c is the counting measure. Note that the family $(\mu_w)_{w>0}$ is locally bounded. Indeed let $C = [-M, M]$ be a closed interval in $\mathbb{R}$ and $M \in \mathbb{N}$. Then the set $C \cap H_w$ contains at most $2[Mw] + 1$ elements. So $\mu_w(C \cap H_w) \leq 2M + 1$, for $w \geq 1$ and hence the assertion follows.

Let now $\Gamma : \mathbb{R} \to \mathbb{R}$ be a summable function and such that the following assumptions hold

(L.1) $\sum_{k=-\infty}^{+\infty} \Gamma(u - k) = 1$ for every $u \in \mathbb{R}$.

(L.2) $m_0(\Gamma) := \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{+\infty} |\Gamma(u - k)| < +\infty$.

Note that from the summability of Γ we deduce the following property

(L.3) For every $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{|s|>M} |\Gamma(s)|ds < \varepsilon.$$

Now, for every $w > 0$, we define

$$\Gamma_w\left(s, \frac{k}{w}\right) = w\Gamma\left(w\left(s - \frac{k}{w}\right)\right) = w\Gamma(ws - k),\ s \in \mathbb{R},\ k \in \mathbb{Z}.$$

We prove that the family of kernels $K_w : G \times H_w \times \mathbb{R} \to \mathbb{R}$, defined by

$$K_w\left(s, \frac{k}{w}, u\right) = w\Gamma(ws - k)u,$$

is singular with $L_w(s, \frac{k}{w}) = w|\Gamma(ws - k)|$.

Firstly, note that the family $(K_w)_{w>0}$ satisfies a strong Lipschitz condition with $\psi_w(u) = u$, for every $u \geq 0$ and $w > 0$. Thus assumption (3.1) is satisfied with $\eta = \varphi$ and $c_w = 0$, for every $w > 0$. Moreover

$$\int_{\mathbb{R}} w|\Gamma(ws - k)|ds = \int_{\mathbb{R}} |\Gamma(u)|du = \|\Gamma\|_1 < +\infty.$$

For a fixed $s \in \mathbb{R}$ and $\delta > 0$, let us put $U_s = [s - \delta, s + \delta]$. Then

$$\int_{H_w\setminus U_s} L_w\left(s, \frac{k}{w}\right) d\mu_w = \sum_{|ws-k|>\delta w} |\Gamma(ws - k)|.$$

By Lemma 1 in [26], assumption (L.2) implies

$$\lim_{R\to+\infty} \sum_{|u-k|>R} |\Gamma(u - k)| = 0,$$

uniformly with respect to $u \in \mathbb{R}$ and so for a fixed $\varepsilon > 0$ there is $R_\varepsilon > 0$ such that

$$\sum_{|u-k|>R_\varepsilon} |\Gamma(u - k)| < \varepsilon,$$

uniformly with respect $u \in \mathbb{R}$. Let $\overline{w} > 0$ such that $w\delta > R_\varepsilon$ for every $w \geq \overline{w}$. Then for $w \geq \overline{w}$,

$$\sum_{|ws-k|>\delta w} |\Gamma(ws - k)| \leq \sum_{|ws-k|>R_\varepsilon} |\Gamma(ws - k)| < \varepsilon,$$

and so the second condition of singularity in satisfied. The third condition immediately follows from **(L.1)**. Finally we show that property $(**)$ is satisfied for the family $(L_w(\cdot, \frac{k}{w}))$. Let $C = [-\delta, \delta]$ be a fixed interval in $\mathbb{R}$ and let $\varepsilon > 0$ be fixed. Let M be the constant in **(L.3)** and let us choose $\widetilde{M} > 0$ such that $\widetilde{M} - \delta > M$. Since $\frac{k}{w} \in [-\delta, \delta]$, we only consider $k \in [-\delta w, \delta w]$. So we have, with $w > 1$,

$$\begin{aligned}\int_{|s|>\widetilde{M}} w|\Gamma(ws-k)|ds &= \int_{-\infty}^{-\widetilde{M}w-k} |\Gamma(u)|du + \int_{\widetilde{M}w-k}^{+\infty} |\Gamma(u)|du \\ &\leq \int_{|u|>(\widetilde{M}-\delta)w} |\Gamma(u)|du \\ &< \int_{|u|>M} |\Gamma(u)|du < \varepsilon.\end{aligned}$$

So the assertion follows taking $B = [-\widetilde{M}, \widetilde{M}]$.

Hence for the family of discrete operators defined by

$$(S_w f)(s) = \sum_{k=-\infty}^{+\infty} \Gamma(ws-k) f\left(\frac{k}{w}\right)$$

we can apply all the previous theory.

Taking $\zeta_w = \|\Gamma\|_1$ for every $w > 0$ and assuming for the sake of simplicity that $\|\Gamma\|_1 = 1$, the class $\Upsilon_{H_w}(P, (\zeta_w)_{w>0})$ now takes the form

$$\begin{aligned}&\Upsilon_{H_w}(P, (\zeta_w)_{w>0}) \\ &\qquad = \left\{ f \in L^\varphi(\mathbb{R}) : \limsup_{w\to+\infty} \frac{1}{w} \sum_{k=-\infty}^{\infty} \varphi\left(\lambda \left| f\left(\frac{k}{w}\right)\right|\right) \leq P \int_{\mathbb{R}} \varphi(\lambda|f(s)|)ds \right\}.\end{aligned}$$

In particular if $f \in BV(\mathbb{R}) \cap E^\varphi(\mathbb{R})$, then $\varphi \circ \lambda|f|$ is also of bounded variation, since φ is locally Lipschitz. Thus, using the results in [19], we have

$$\lim_{w\to+\infty} \frac{1}{w} \sum_{k=-\infty}^{\infty} \varphi\left(\lambda \left| f\left(\frac{k}{w}\right)\right|\right) = \int_{\mathbb{R}} \varphi(\lambda|f(s)|)ds,$$

see also [9] and [10]. Now, if $f \in BV(\mathbb{R}) \cap E^\varphi(\mathbb{R})$, taking $\mathcal{F} = C_c^\infty(\mathbb{R})$, we have $f - \mathcal{F} \subset BV(\mathbb{R}) \cap E^\varphi(\mathbb{R})$ and since $\overline{\mathcal{F}}_\varphi = L^\varphi(\mathbb{R})$, (see [18], [2]), we obtain the following

Corollary 4.1. *Let $f \in BV(\mathbb{R}) \cap E^\varphi(\mathbb{R})$. Then there exists $\alpha > 0$ such that*

$$\lim_{w\to+\infty} \rho^\varphi_{\mathbb{R}}[\alpha(S_w f - f)] = 0.$$

If we take $\varphi(u) = |u|^p$, we obtain a convergence result in L^p-spaces.

Thus we obtain a non local modular approximation theorem in Orlicz spaces for the generalized sampling series of a function f and this improves corresponding results given in [7] and [27] for kernels with compact support.

The above results can be extended also to the nonlinear discrete operators of the form

$$(T_w f)(s) = \sum_{k=-\infty}^{+\infty} K\left(ws-k, f\left(\frac{k}{w}\right)\right),$$

where the corresponding functions L_w satisfy the previous conditions (see [27], [23], [6] and [1]).

As a second example, let us consider $G = [0, +\infty[$ with Lebesgue measure and for every $n \in \mathbb{N}$ we take $H_n = \frac{1}{n}\mathbb{N}_0 = \{\frac{k}{n},\ k = 0, 1, \dots\}$ endowed with the measure $\frac{1}{n}\mu_c$. Let us consider the class of the Szász-Mirak'jan operators given by

$$(S_n f)(s) = \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right)\sigma_{k,n}(s),\ s \geq 0,$$

where

$$\sigma_{k,n}(s) = e^{-ns}\frac{(ns)^k}{k!},\ s \geq 0.$$

These operators are generated by the family of functions defined by

$$L_n\left(s, \frac{k}{n}\right) = nL\left(ns, n\frac{k}{n}\right) = nL(ns, k),$$

where

$$L(s, k) = e^{-s}\frac{s^k}{k!}.$$

As shown in [4], the family of kernels $K_n : G \times H_n \times \mathbb{R} \to \mathbb{R}$ defined by

$$K_n\left(s, \frac{k}{n}, u\right) = L_n\left(s, \frac{k}{n}\right)u$$

satisfies all the singularity assumptions and $\zeta_n = 1$ for every $n \in \mathbb{N}$. Now we show that the family $(L_n)_{n\in\mathbb{N}}$ satisfies $(*,*)$. Firstly note that without loss of generality we can take $C = [0, \delta]$, with $\delta > 1$ a positive integer. Let $B = [0, M]$ with $M > 1$ and $\frac{M}{\log M+1} > \delta$. We have

$$\int_{G\setminus B} L_n\left(s, \frac{k}{n}\right) ds = \int_M^{+\infty} ne^{-ns}\frac{(sn)^k}{k!}ds := I[n, k],$$

where $k \in [0, n\delta]$. By using elementary calculus and a recursion method, we have

$$I[n, k] = e^{-nM}\sum_{j=0}^{k}\frac{(nM)^j}{j!}$$

and so

$$\begin{aligned} I[n, k] &\leq e^{-nM}\sum_{j=0}^{n\delta}\frac{(nM)^j}{j!} \\ &\leq \frac{M^{n\delta}e^{-nM}}{\delta}\sum_{j=0}^{+\infty}\frac{(n\delta)^j}{j!} \\ &= \frac{M^{n\delta}}{\delta}e^{-n(M-\delta)} = \frac{1}{\delta}\left(\frac{M^\delta}{e^{M-\delta}}\right)^n. \end{aligned}$$

Since $M^\delta < e^{M-\delta}$, $(*,*)$ follows.

Note that the family of functions $(L_n)_{n\in\mathbb{N}}$ also satisfies property $(*)$ for every $\beta > 0$.

Finally we obtain the following convergence theorem for the Szász-Mirak'jan operators in Orlicz spaces generated by a convex φ-function φ.

Corollary 4.2. *Let $f \in BV(\mathbb{R}_0^+) \cap E^\varphi(\mathbb{R}_0^+)$. Then there exists $\alpha > 0$ such that*

$$\lim_{n\to+\infty}\rho^\varphi_{\mathbb{R}_0^+}[\alpha(S_n f - f)] = 0.$$

Remark 4.3. Note that, using the characterization in [19], Corollaries 4.1 and 4.2 are still true under the weaker assumption that the function f, (or the function $\varphi \circ \lambda|f|$), is locally Riemann integrable and of bounded "coarse" variation on $\mathbb{R}_0^+$. This notion was introduced in [19] in connection with "simply integrable functions" and it is meaningful if the domain of f is unbounded, since for bounded domains it is equivalent just to boundedness of the function f.

More generally, let us consider a (increasing) sequence $(t_n)_{n\in\mathbb{Z}}$ of real numbers such that $|t_n - t_{n-1}| \geq r$, for every $n \in \mathbb{Z}$ and an absolute positive constant r. Let $\mathcal{T} = \{t_n : n \in \mathbb{Z}\}$. Let us put $G = \mathbb{R}$ and $H_w = \frac{1}{w}\mathcal{T}$. We provide again G with the Lebesgue measure and we put $\mu_w = \frac{1}{w}\mu_c$. Let now $\Gamma : \mathbb{R} \times \mathcal{T} \to \mathbb{R}$ be a function such that $\Gamma \in L^1(\mathbb{R})$ with respect to $u \in \mathbb{R}$ and

(Γ.1) $\sum_{n=-\infty}^{+\infty} \Gamma(u, t_n) = 1$ for every $u \in \mathbb{R}$.

(Γ.2) $m_1(\Gamma) := \sup_{u\in\mathbb{R}} \sum_{n=-\infty}^{+\infty} |\Gamma(u,t_n)|\{1 + |u - t_n|\} < +\infty$.

(Γ.3) For every $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{|s|>M} |\Gamma(s,t_n)|ds < \varepsilon,$$

for every $t_n \in \mathcal{T}$.

Now, for every $w > 0$ we define the family of functions $\Gamma_w : \mathbb{R} \times H_w \to \mathbb{R}$ on putting

$$\Gamma_w\left(s, \frac{t_n}{w}\right) = w\Gamma(ws, t_n).$$

As before, we can see that the family of kernels $K_w : \mathbb{R} \times H_w \times \mathbb{R} \to \mathbb{R}$ defined by

$$K_w\left(s, \frac{t_n}{w}, u\right) = w\Gamma(ws, t_n)u$$

satisfies a Lipschitz condition with $\psi_w(u) = u$ and $L_w\left(s, \frac{t_n}{w}\right) = w|\Gamma(ws, t_n)|$ for every $u \geq 0$, and it is singular.

Note that the property $(*, *)$ is a consequence of **(Γ.3)** by using the same arguments as the first example.

Thus the previous theory can be applied to Kramer type operators of the form

$$(T_w f)(s) = \sum_{n\in\boldsymbol{Z}} \Gamma_w\left(s, \frac{t_n}{w}\right) f\left(\frac{t_n}{w}\right)$$

and their nonlinear versions of the form

$$(T_w f)(s) = \sum_{n\in\mathbb{Z}} \Gamma_w\left(s, \frac{t_n}{w}, f\left(\frac{t_n}{w}\right)\right).$$

Finally, some multidimensional versions of discrete operators are included in our general approach. For example let us consider the multivariate generalized sampling operator defined by (see [12])

$$(S_{\mathtt{w}} f)(\mathtt{s}) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{\mathtt{k}}{\mathtt{w}}\right)\Gamma(\mathtt{ws} - \mathtt{k}),$$

where $\mathtt{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$, $\mathtt{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$, $\mathtt{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $\frac{\mathtt{k}}{\mathtt{w}} = (k_1/w_1, \ldots, k_n/w_n)$, $\mathtt{ws} = (w_1 s_1, \ldots, w_n s_n)$ and $f : \mathbb{R}^n \to \mathbb{R}$. Here the kernel Γ is given, for example, by a "box" kernel

$$\Gamma(\mathtt{s}) = \prod_{j=1}^{n} \Gamma_j(s_j),$$

where Γ_j is a one-dimensional kernel satisfying the above assumptions.

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